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A family of stochastic models for two-particle dispersion in isotropic homogeneous stationary turbulence

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A family of Lagrangian stochastic models for the joint motion of particle pairs in isotropic homogeneous stationary turbulence is considered. The Markov assumption and well-mixed criterion of Thomson (1990) are used, and the models have quadratic-form functions of velocity for the particle accelerations. Two constraints are derived which formally require that the correct one-particle statistics are obtained by the models. These constraints involve the Eulerian expectation of the 'acceleration' of a fluid particle with conditioned instantaneous velocity, given either at the particle, or at some other particle's position. The Navier–Stokes equations, with Gaussian Eulerian probability distributions, are shown to give quadratic-form conditional accelerations, and models which satisfy these two constraints are found. Dispersion calculations show that the constraints do not always guarantee good one-particle statistics, but it is possible to select a constrained model that does. Thomson's model has good one-particle statistics, but is shown to have unphysical conditional accelerations. Comparisons of relative dispersion for the models are made.

1. Introduction

The Lagrangian statistical description of turbulent dispersion dates back to Taylor (1921). Batchelor (1949, 1952) has extended those initial ideas, and related fluid particle dispersion to passive-tracer concentrations (see §3). Obukhov (1959) was the first to propose a stochastic differential (Langevin) equation to model dispersion of one particle in a homogeneous turbulent flow. Subsequently, many authors have sought to extend the one-particle stochastic-equation modelling technique to more complex flows (Sawford 1985; Sawford & Guest 1988) or to develop a two-particle model (Novikov 1963; Durbin 1980; Sawford & Hunt 1986; and Kaplan & Dinar 1989). Pope (1987) also gives an account of stochastic modelling of particle trajectories. Many of these attempts have been of an *ad hoc* or otherwise inadequate nature. Recently, Thomson (1987, 1990) and Pope (1994*a*, *b*) have provided a much more comprehensive and rigorous approach to both one-particle and two-particle modelling. In this paper we propose to continue the development of Thomson's (1990) approach regarding the two-particle dispersion problem.

The models mentioned above have been developed explicitly for idealized forms of turbulence and have been only partially verified for real flows approximating these idealized situations. For example, Thomson (1990) developed a two-particle model for isotropic homogeneous decaying turbulence, generated concentration statistics numerically and compared these results with the wind-tunnel data of Warhaft & Lumley (1978). The comparisons are encouraging but do not test the model in detail.

For relative dispersion there are few other data with which to compare stochastic-model predictions directly, particularly in the extremely large Reynolds number range where such models are formally applicable. There are some direct numerical simulations of turbulence (Yeung & Pope 1989) from which Lagrangian quantities may be calculated, but these calculations are not yet of large enough Reynolds number for the comparisons to be unambiguous. Similarly, kinematic simulation of flow fields (Fung *et al.* 1992) is not yet sufficiently rigorous to clarify many of the questions we would like to answer. Because of this lack of suitable Lagrangian data for model verification, it is important to be as rigorous and objective as possible in the development of models and to incorporate known physical and mathematical properties properly.

It is natural to begin with the simplest case and therefore we consider isotropic, homogeneous and stationary turbulence. It is also valid to ignore intermittency for the level of discussion attempted here (Borgas & Sawford 1991, 1994).

The pivotal underlying assumption for our stochastic modelling is that in the velocity–displacement phase space, in which continuous trajectories describe the history of one or more particles, the random process which makes up that evolution is approximately Markovian (this approximation becomes better as the Reynolds number increases). Thus the future change in velocity is dependent solely upon the present velocity and position and not upon its history. This is a difficult proposition to prove from first principles. An elaborate asymptotic analysis by Borgas & Sawford (1991) and Borgas (1991) supports this view. However, only the necessary (but not sufficient) condition that accelerations decorrelate rapidly with time, on a timescale that diminishes with increasing Reynolds number, is generally accepted. Nevertheless, we assume that the process is approximately Markovian. Sawford & Borgas (1994) and Borgas & Sawford (1994) consider in more detail non-Markovian aspects of the Lagrangian velocity–displacement properties of turbulence.

Once the Markovian property is accepted, a particle at position $\boldsymbol{x}$ with velocity $\boldsymbol{u}$ at time t changes its position and velocity by increments which are solely functions of $\boldsymbol{x}$, $\boldsymbol{u}$ and t and some independent random increment (see §2). Choosing a model consists of specifying 'recipes' for those functions. The approach of Thomson (1987, 1990) is to specify the models in terms of given Eulerian velocity statistics, via the probability distribution of the random velocity at $\boldsymbol{x}$ (or velocities at two points for two-particle dispersion). Typically this probability distribution is taken to be Gaussian, or perhaps a sum of several (suitably normalized) Gaussian distributions. Thus the Eulerian statistics are specified in practice by one- and two-point velocity structure functions which are common turbulence-field diagnostics.

The importance of Thomson's approach is that it ensures that when the material distribution is uniform, the model equations are satisfied by Eulerian flow statistics. This also ensures that the model does not artificially un-mix material and (in the two-particle case) does not generate spurious concentration variance in the absence of gradients in the mean concentration.

A further assumption that has been implicit in the modelling is that the energy-dissipation rate per unit volume, ϵ, which is a random variable locally (the spatial average of the energy-dissipation rate is a fundamental fixed parameter for any steady-state situation) is non-intermittent (Monin & Yaglom 1975). Novikov (1989, 1990), Yeung & Pope (1989), Pope & Chen (1990), and Borgas (1991, 1993) consider the implications intermittency has for Lagrangian statistics, and Borgas & Sawford (1994) analyse the impact upon stochastic models and the Markov property. In particular, it is shown that for one-particle dispersion the effects of intermittency make the

displacement p.d.f. non-Gaussian, but that this effect is so slight that it could not be detected in experiments. Overall, the results from non-intermittent Lagrangian stochastic models, which are employed for the remainder of this paper, are largely unaffected by the small-scale intermittency of turbulence, and may be regarded as capturing the leading-order contribution to dispersion (especially for the first few moments).

Despite the objectivity and relative rigour of Thomson's approach there still remain problems which we attempt to address in this paper. Perhaps the major problem is that Thomson's procedure does not yield a unique result, i.e. there may be many models which are consistent with the specified Eulerian statistics and at present there is no objective way to choose between these models. This was confirmed by numerical calculations for the one-particle case in inhomogeneous turbulence by Sawford & Guest (1988). Here we show that this non-uniqueness is non-trivial for the two-particle problem even for the simplest possible turbulence field: homogeneous and isotropic turbulence. In particular, in §4 we obtain specific forms of the model under the assumption of Gaussian Eulerian velocity statistics. We demonstrate, both mathematically and numerically, that the specification of the models is incomplete and that they are therefore not unique. In §5 the non-uniqueness of the models is considered in greater detail and a general family of models is found. One of these is the model presented by Thomson (1990).

A second problem with the model presented by Thomson (1990) is that one-particle statistics (obtained by averaging over the second particle) are not consistent with those obtained from the one-particle model for homogeneous isotropic turbulence (which we prove is unique). Whether or not these differences are significant in practical terms, such a situation is clearly fundamentally wrong. In §6 we formulate a principle, called 'two-to-one reduction' requiring consistency between these two sets of one-particle statistics. We are thus able to generate further mathematical constraints on the two-particle models and use these constraints to distinguish objectively among the various possible models. These constraints are derived in general, in particular avoiding the restriction to Gaussian Eulerian velocity statistics; however, the precise implications of Gaussian statistics are considered in greatest detail.

We implement the two-to-one reduction and test its efficacy by numerical calculation in §7. Surprisingly, although Thomson's model fails to satisfy our constraints it performs well in the numerical tests (prediction of the one-particle velocity variance and dispersion) which we apply. However, a new model is found which is formally better and which satisfies some dynamical constraints (conditional pressure gradients) imposed by the Navier Stokes equations and the form of the Eulerian statistics. This model is the first two-particle stochastic model to make use of this dynamical information, and differs from Pope's (1985) application of such ideas. Furthermore, the dynamical constraints show that the family of models that we consider explicitly is the natural family for Gaussian Eulerian statistics. While this family, and Gaussian Eulerian statistics, are primarily used for mathematical convenience, the comparison with the most significant historical benchmark (Thomson 1990) also requires this approach.

2. Background on one-particle stochastic equations

According to the Markov assumption, trajectories of independent fluid particles in a turbulent flow can be modelled by the stochastic differential equations (Thomson 1987, 1990)

$$\mathrm{d}u_i = a_i(\boldsymbol{u}, \boldsymbol{x}, t)\,\mathrm{d}t + (C_0\epsilon)^{1/2}\,\mathrm{d}W_i \tag{2.1}$$

and
$$\mathrm{d}x_i = u_i\,\mathrm{d}t, \tag{2.2}$$

where $\boldsymbol{u}$ is the particle velocity, $\boldsymbol{x}$ is the particle displacement and $\mathrm{d}\boldsymbol{W}$ is white noise with correlation

$$\langle \mathrm{d}W_i(t)\,\mathrm{d}W_j(t+\tau)\rangle = \delta_{ij}\,\delta(\tau)\,\mathrm{d}t\,\mathrm{d}(t+\tau). \tag{2.3}$$

The coefficient $(C_0\epsilon)^{\frac{1}{2}}$ ensures that the Lagrangian velocity structure function satisfies Kolmogorov's scaling in the inertial range, i.e.

$$\langle \mathrm{d}u_i(t)\,\mathrm{d}u_j(t)\rangle = \delta_{ij}\,C_0\,\epsilon\,\mathrm{d}t.$$

ϵ is a parameter which describes the mean rate of energy dissipation in the flow while C_0 is a universal constant so that, in principle, (2.1) is applicable to all forms of turbulence for one empirically determined value of C_0. The undetermined function $\boldsymbol{a}$ may depend upon ϵ and σ (the r.m.s. turbulent velocity fluctuation) as well as the arguments shown explicitly. To model turbulent dispersion correctly, the proper choice for $\boldsymbol{a}$ must be made.

Equations (2.1) and (2.2) are equivalent to the Fokker Planck equation:

$$\frac{\partial P_L}{\partial t} + \boldsymbol{u}_j\frac{\partial P_L}{\partial \boldsymbol{x}_j} = -\frac{\partial}{\partial \boldsymbol{u}_j}(a_j P_L) + \tfrac{1}{2}C_0\,\epsilon\frac{\partial^2 P_L}{\partial \boldsymbol{u}_j\,\partial \boldsymbol{u}_j}, \tag{2.4}$$

where $P_L = P_L(\boldsymbol{u}, \boldsymbol{x}, t; \boldsymbol{u}_0, \boldsymbol{x}_0, t_0)$ is the joint probability density for the velocity and position of a particle given its velocity, $\boldsymbol{u}_0$, and position, $\boldsymbol{x}_0$, at time t_0. We have assumed that the fluid is incompressible, i.e. $\partial u_i/\partial x_i = 0$, but compressibility and variable density can be accommodated (Thomson 1987, 1990). The subscript L indicates that the probability is Lagrangian. Alternatively, if we take an ensemble average of P_L over an Eulerian (unbiased) distribution of initial states, the Eulerian probability distribution of velocities, $P_E(\boldsymbol{u}; \boldsymbol{x}, t)$, results, i.e.

$$P_E(\boldsymbol{u}; \boldsymbol{x}, t) = \int P_L(\boldsymbol{u}, \boldsymbol{x}, t; \boldsymbol{u}_0, \boldsymbol{x}_0, t_0)\,P_E(\boldsymbol{u}_0; \boldsymbol{x}_0, t_0)\,\mathrm{d}^3\boldsymbol{u}_0\,\mathrm{d}^3\boldsymbol{x}_0.$$

Since the Fokker–Planck equation (2.4) is linear in P_L, it is also satisfied by P_E. Now P_E may be considered as a prescribed property of the turbulence, and therefore constrains $\boldsymbol{a}$ through (2.4). For example, suppose that the turbulence field is homogeneous, isotropic and stationary, then P_E is independent of $\boldsymbol{x}$ and t. Equation (2.4) (with P_E) can then be solved for a_i to give

$$a_i = \tfrac{1}{2}C_0\,\epsilon\frac{1}{P_E}\frac{\partial P_E}{\partial u_i} + \phi_i. \tag{2.5}$$

Equation (2.5) has been described as the 'well-mixed' criterion by Thomson (1987, 1990), and is absolutely essential for the successful simulation of ensembles of particle trajectories. It ensures, for example, that an initially well-mixed distribution of material is not 'un-mixed' by the action of the turbulence. The vector $\boldsymbol{\phi}$, where

$$\frac{\partial}{\partial \boldsymbol{u}_i}(P_E\,\phi_i) = 0,$$

is included for completeness and shows that (2.5) is, in general, not a unique representation of $\boldsymbol{a}$.

However, if we calculate the ensemble average of the conditional 'acceleration' of a particle (for all flows with $\boldsymbol{u} = \boldsymbol{v}$, a given constant velocity vector at a given position $\boldsymbol{x}$ at a given time t) then it follows from (2.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u_i(t)\,|\boldsymbol{v}(t')\rangle|_{t=t'} = a_i(\boldsymbol{v}, \boldsymbol{x}, t).$$

It follows from isotropy and homogeneity that

$$a_i(\boldsymbol{v}) = V(v)\, v_i,$$

for some function $V(v)$, where $v^2 = v_j v_j$. This form ensures that the mean acceleration vector is independent of rotations and reflections of coordinate space (isotropy) and translations of the coordinate space origin (homogeneity). Because of (2.5), we must also have

$$P_E\, \phi_i = \psi(v)\, v_i$$

and therefore that

$$\psi' + 3v^{-1}\psi = 0;$$

thus a definite form for ϕ_i emerges:

$$\phi_i(\boldsymbol{v}) = P_E^{-1}\, c v^{-3} v_i. \tag{2.6}$$

It is straightforward to see that only when the constant c vanishes can (2.5) and (2.6) represent a suitable form (consider either $v \to 0$ or ∞).† Thus a relatively simple determination of the appropriate stochastic equation is possible in this case. The further assumption that

$$P_E = (2\pi\sigma^2)^{-3/2} \exp\left(-\frac{1}{2}\frac{v^2}{\sigma^2}\right) \tag{2.7}$$

finally gives the appropriate version of (2.1) as

$$\mathrm{d}u_i = -(\tfrac{1}{2}C_0\, \epsilon\sigma^{-2})\, u_i\, \mathrm{d}t + (C_0\, \epsilon)^{1/2}\, \mathrm{d}W_i. \tag{2.8}$$

Note that the non-stationary case also yields a unique specification of a_i provided that the flow is both isotropic and homogeneous.

When solving (2.8), the usual initial condition is that the velocity $\boldsymbol{u}_0$ is chosen randomly so that the probability distribution is $P_E(\boldsymbol{u}_0)$. It is possible then to solve the linear system (2.8) and (2.2) exactly, to give the Lagrangian probability distribution for position (averaged over all initial velocities)

$$P_L = \frac{(2\pi)^{-3/2}}{\mathscr{D}^{1/2}} \exp\left(-\tfrac{1}{2}(x_i - x_i(0))\, \mathscr{D}_{ij}(x_j - x_j(0))\right),$$

where

$$\mathscr{D}_{ij}^{-1} = 2\sigma^2 t_L^2 \left[\exp\left(-\frac{t}{t_L}\right) + \frac{t}{t_L} - 1\right]\delta_{ij}, \tag{2.9}$$

with $t_L = 2\sigma^2/(C_0\, \epsilon)$ and $\mathscr{D} = |\mathscr{D}|$, the determinant of $\mathscr{D}$.

† If P_E falls off algebraically for large v then it must do so faster than v^{-5} so that the kinetic energy is bounded. However, in such situations a_i would increase faster than v^2 with large v. Gardiner (1983) points out that such models have the velocities accelerating to infinity in finite time which is unacceptable.

Equation (2.9) is a well-known result of turbulence theory (Taylor 1921), and has been obtained independently of the formulation presented here. The point to note is that isotropy, homogeneity and knowledge of the Eulerian P_E, fully determine $\boldsymbol{a}$, and therefore allow the calculation of the Lagrangian properties and thus dispersion. However, while this happens in this special case, it is not generally so for inhomogeneous turbulence. Further work is required for the determination of $\boldsymbol{\phi}$ in more complex situations and this is yet to be done.

3. Two-particle statistics

We are particularly interested in the generalization of the system in §2 to involve joint trajectories of particle pairs. The Lagrangian statistics allow higher-order moments of dispersion statistics to be determined. For example, if $\mathscr{S}(\boldsymbol{x}_0)$ gives the initial concentration† of some passive marker with very small molecular diffusivity then for subsequent times the concentration-field structure is given by

$$\langle \mathscr{C}(\boldsymbol{x})\mathscr{C}(\boldsymbol{x}+\boldsymbol{\Delta})\rangle = \int \mathscr{S}(\boldsymbol{x}_0)\,\mathscr{S}(\boldsymbol{x}_0')\,P_L(\boldsymbol{x},\boldsymbol{x}+\boldsymbol{\Delta},t;\boldsymbol{x}_0,\boldsymbol{x}_0',t_0)\,\mathrm{d}^3\boldsymbol{x}_0\,\mathrm{d}^3\boldsymbol{x}_0',$$

where P_L is shown by its arguments to be the joint probability distribution for the position of both particles. Consequently, the mean-square concentration is

$$\langle \mathscr{C}^2(\boldsymbol{x})\rangle = \int \mathscr{S}(\boldsymbol{x}_0)\,\mathscr{S}(\boldsymbol{x}_0')\,P_L(\boldsymbol{x},\boldsymbol{x},t;\boldsymbol{x}_0,\boldsymbol{x}_0',t_0)\,\mathrm{d}^3\boldsymbol{x}_0\,\mathrm{d}^3\boldsymbol{x}_0'.$$

The validity of this formula for mean-square concentration is discussed by Durbin (1980) and more recently by Thomson (1990) and is based on the limit in which the molecular diffusivity of the marker vanishes so that the dispersion due to turbulent advection dominates. This limiting procedure also allows us to neglect explicit viscous effects, i.e. we envisage a formal limit of large Reynolds number, $Re\to\infty$, and we restrict our two-particle models to particle separations greater than the Kolmogorov microscale, $\eta = Re^{-3/4}L$ (Borgas & Sawford 1991), where L is a lengthscale of the turbulence, say σ^3/ϵ.

For the two-particle problem we hope to emulate the results of §2, i.e. find appropriate stochastic equations for trajectories based on Eulerian statistics and then to provide solutions for P_L. The two-particle problem is conveniently represented using six-dimensional vectors, $\boldsymbol{u}$ and $\boldsymbol{x}$, where the first three components refer to particle one, the latter three to particle two. It is then possible to represent the Markov process exactly as in (2.1) and (2.2). The Fokker–Planck equation also has the same form as (2.4). Specific differences emerge because the two-point Eulerian statistics, even for stationary, homogeneous and isotropic turbulence, depend on the separation of the fixed sampling points. Thus explicit dependence on the spatial coordinates is introduced, and we now write

$$P_E(\boldsymbol{u};\boldsymbol{x}) = P_E(\boldsymbol{u};\boldsymbol{\Delta}),$$

where $\boldsymbol{\Delta} = \boldsymbol{x}^{(2)} - \boldsymbol{x}^{(1)}$ is a three-dimensional vector giving the separation of the points $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ in physical space. To avoid ambiguity where the same symbols are used to represent both three- and six-dimensional vectors, the former are often written with superscript particle labels (as done above for particles 'one' and 'two'). In other cases, the dimensionality will be clear from the context.

† Concentration is defined as the 'amount' of material within some fixed small volume $\delta^3\boldsymbol{x}$ surrounding the point $\boldsymbol{x}$. For any finite volume there are effectivcly many fluid particles contained within, i.e. the test volume for defining concentration is assumed much larger than the idealized fluid particles whose trajectories we simulate.

That P_E depends non-trivially upon separation can be seen by considering the Eulerian velocity covariances:

$$\langle u_i u_j \rangle = \begin{pmatrix} \langle u_{i'}^{(1)} u_{j'}^{(1)} \rangle & \langle u_{i'}^{(1)} u_{j''}^{(2)} \rangle \\ \langle u_{i''}^{(2)} u_{j'}^{(1)} \rangle & \langle u_{i''}^{(2)} u_{j''}^{(2)} \rangle \end{pmatrix}, \tag{3.1a}$$

where
$$\langle u_{i'}^{(1)} u_{j'}^{(1)} \rangle = \langle u_{i''}^{(2)} u_{j''}^{(2)} \rangle = \sigma^2 \delta_{i'j'} \tag{3.1b}$$

and
$$\langle u_{i'}^{(1)} u_{j'}^{(2)} \rangle = \langle u_{i''}^{(2)} u_{j'}^{(1)} \rangle = \sigma^2 \rho_{i'j'}(\Delta), \tag{3.1c}$$

with
$$\rho_{i'j'}(\Delta) = \frac{f(\Delta) - g(\Delta)}{\Delta^2} \Delta_{i'} \Delta_{j'} + g(\Delta)\, \delta_{i'j'}. \tag{3.1d}$$

The notation developed above will also be useful in other contexts; since the higher-order vector spaces we consider are integer multiples of three, it is possible to represent an mth-order tensor in $n \times 3$ dimensions as an mth-order block tensor in n dimensions whose n^m elements are all three-dimensional tensors. Note that while i and j are indices from the set $\{1, 2, \ldots, 6\}$, the indices with $'$ or $''$ superscripts are either 1, 2 or 3: defined as $i = i'$ if $i \leqslant 3$ or $i'' = i - 3$ if $i > 3$ and so forth.

The functions $f(\Delta)$ and $g(\Delta)$ are known to be non-trivial (Batchelor 1953) but have been measured experimentally and are therefore, for our purposes, supposed known. Note that continuity requires that

$$g(\Delta) = f(\Delta) + \tfrac{1}{2}\Delta \frac{\mathrm{d}f}{\mathrm{d}\Delta}. \tag{3.2}$$

Furthermore, for the purposes of calculation we will use the simple parameterized form

$$f(\Delta) = 1 - \left(\frac{\Delta^2}{\Delta^2 + \mathscr{L}^2} \right)^{1/3} \tag{3.3}$$

for some lengthscale $\mathscr{L}$. This has many useful properties (which are discussed below) and has been used extensively in the past (Durbin 1980; Thomson 1990). It is possible to estimate $\mathscr{L}$ by following Thomson (1990): calculating the integral lengthscale,

$$\mathscr{L}_f = \int_0^\infty f(\Delta)\, \mathrm{d}\Delta = \frac{\Gamma(\frac{5}{6})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3})} \mathscr{L} = 0.747 \mathscr{L},$$

and using the empirical fact (Townsend 1976, pp. 61) that $\mathscr{L}_f \approx 0.8\sigma^3\epsilon^{-1}$ suggests that $\mathscr{L} \approx \sigma^3\epsilon = L$ (say). In the inertial sub-range ($\Delta \ll L$), (3.3) takes the form

$$f(\Delta) \approx 1 - \left(\frac{\Delta}{\mathscr{L}} \right)^{2/3} = 1 - \tfrac{1}{2} C \sigma^{-2} \epsilon^{2/3} \Delta^{2/3},$$

where the Kolmogorov constant C is given by $2(L/\mathscr{L})^{2/3} \approx 2$, in accord with measurement (Monin & Yaglom 1975, pp. 485).

An implicit property of (3.1 *a*) and (3.1 *b*) is the two-to-one reduction which says that ensemble averages of quantities dependent solely on particle one should be equivalent to the one-particle results briefly outlined in §2. Therefore the ensemble average of $u_{i'}^{(1)} u_{j'}^{(1)}$, as given in (3.1 *a*), is independent of Δ and follows directly from (2.7). More extensive use of the two-to-one reduction will be outlined below.

Because of the dependence of P_E on the separation Δ, ϕ must satisfy the Eulerian constraint

$$\frac{\partial}{\partial u_i}(P_E \phi_i) = -u_j \frac{\partial P_E}{\partial x_j} = -\left(u_{j'}^{(2)} \frac{\partial P_E}{\partial \Delta_{j'}} - u_{j'}^{(1)} \frac{\partial P_E}{\partial \Delta_{j'}} \right). \tag{3.4}$$

However, $\boldsymbol{\phi}$ is still not determined uniquely since an arbitrary divergence-free vector in six-dimensional $\boldsymbol{u}$-space may be added to $P_E\boldsymbol{\phi}$ without affecting (3.4). The two-particle stochastic equation for velocity increments now has the form

$$\left.\begin{aligned} \mathrm{d}u_{i'}^{(1)} &= -\tfrac{1}{2}C_0\epsilon\frac{\partial}{\partial u_{i'}}\log\left(P_E(\boldsymbol{u},\boldsymbol{\Delta})\right)\mathrm{d}t+\phi_{i'}(\boldsymbol{u},\boldsymbol{\Delta})\,\mathrm{d}t+(C_0\epsilon)^{1/2}\,\mathrm{d}W_{i'}^{(1)},\\ \mathrm{d}u_{i''}^{(2)} &= -\tfrac{1}{2}C_0\epsilon\frac{\partial}{\partial u_{i''}}\log\left(P_E(\boldsymbol{u},\boldsymbol{\Delta})\right)\mathrm{d}t+\phi_{i''}(\boldsymbol{u},\boldsymbol{\Delta})\,\mathrm{d}t+(C_0\epsilon)^{1/2}\,\mathrm{d}W_{i''}^{(2)}, \end{aligned}\right\} \tag{3.5}$$

which differs from the former Langevin equation because of the $\boldsymbol{\Delta}$-dependence of the coefficient of the terms linear in velocity, because of the cross-dependence of particle one's velocity increment on the velocity of particle two, and because of the non-trivial (and non-unique) $\boldsymbol{\phi}$-function.

Invoking isotropy, homogeneity etc. is of less utility in the two-particle case because ensemble averages generally depend on a trio of three-dimensional vectors: $\boldsymbol{u}^{(1)}$, $\boldsymbol{u}^{(2)}$ and $\boldsymbol{\Delta}$. From these a total of six independent scalars can be formed, and the averages will generally depend on all of them. Thus the general form of $a_{i'}^{(1)}$ is

$$a_{i'}^{(1)} = \mu_1 u_{i'}^{(1)}+\mu_2 u_{i'}^{(2)}+\mu_3\Delta_{i'}, \tag{3.6}$$

where $$\mu_n = \mu_n(\Delta, u_{i'}^{(1)}u_{i'}^{(1)}, u_{i''}^{(2)}u_{i''}^{(2)}, u_{i'}^{(1)}u_{i''}^{(2)}, \Delta_{i'}u_{i'}^{(1)}, \Delta_{i'}u_{i''}^{(2)})$$

for each n. It will be shown that the imposition of isotropy through forms like (3.6) is not sufficient to define a_i uniquely. There is little to be gained by introducing these reduced forms at this stage and we continue for the present with the general tensor notation.

Note that (3.5) and (3.6) encapsulate the dependence of the motion of particle one on that of particle two. We show in §6 and Appendix A that (3.6) is consistent with the mean acceleration calculated from the Navier–Stokes equation. Alternative two-particle models (e.g. Kaplan & Dinar 1989; Novikov 1963), in which $a_{i'}^{(1)}$ is required to be independent of $u_{i''}^{(2)}$ and $\Delta_{i'}$, are consistent with this trivial mean acceleration only when the particle motions are independent of each other.

A simple assumption, which permits some general progress, is that the two-point Eulerian probability distribution is Gaussian; therefore P_E is fully determined by the velocity covariances since the average velocities all vanish. While it is certainly the case that the distribution is non-Gaussian in real turbulence, with the relative velocity field skewed for example, for our purposes the simple model is adequate. This is because the assumption is primarily used to derive an example of a stochastic equation to compare with Thomson's (1990) equation, where the same conditions apply. In addition, results are derived later for general forms of P_E so that the assumption does not limit us at all, although whenever a definite example is discussed we restrict ourselves to the Gaussian P_E case.

Supposing that the velocity covariance is written

$$\langle u_i u_j\rangle = \sigma^2\lambda_{ij}^{-1},$$

with the superscript minus one indicating the elements of the inverse of $\boldsymbol{\lambda}$, it follows that

$$P_E = \lambda^{1/2}(2\pi\sigma)^{-3}\exp\left(-\tfrac{1}{2}\sigma^{-2}u_i\lambda_{ij}u_j\right), \tag{3.7}$$

where the index summation (both i and j) runs from 1 to 6 and $\lambda = |\boldsymbol{\lambda}|$ (the determinant of $\boldsymbol{\lambda}$). In the sense of (3.1) we write

$$\boldsymbol{\lambda}^{-1} = \begin{pmatrix} \boldsymbol{\delta} & \boldsymbol{\rho} \\ \boldsymbol{\rho} & \boldsymbol{\delta} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\beta} & \boldsymbol{\alpha} \end{pmatrix}$$

where care is needed because there is no explicit differentiation between tensors in three- or six-dimensional space in this notation. The three-dimensional tensors are interrelated by

$$\boldsymbol{\alpha} = (\boldsymbol{\delta} - \boldsymbol{\rho}\boldsymbol{\rho})^{-1} \quad \text{and} \quad \boldsymbol{\beta} = -\boldsymbol{\rho}(\boldsymbol{\delta} - \boldsymbol{\rho}\boldsymbol{\rho})^{-1} = -\boldsymbol{\rho}\boldsymbol{\alpha}.$$

The block structure of the six-dimensional forms demonstrates a further necessary property of the two-particle modelling, which is that the labelling of particles is essentially arbitrary so that calling one particle 'one' and the other particle 'two' is equivalent to the converse labelling; in other words, the physics is unaltered by the choice of labelling. This is the case for the tensors above, as is indicated by the symmetry of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^{-1}$, but this principle must be applied more generally to the model, particularly to the vector $\boldsymbol{a}$. For the latter we must therefore have

$$\boldsymbol{a}^{(1)}(\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \boldsymbol{\Delta}) = \boldsymbol{a}^{(2)}(\boldsymbol{u}^{(2)}, \boldsymbol{u}^{(1)}, -\boldsymbol{\Delta}). \tag{3.8}$$

4. Quadratic-form models

We now provide some examples of stochastic models satisfying Thomson's (1990) well-mixed criterion for the case that P_E is Gaussian. This serves a particularly important purpose because we are able to show explicitly that for given P_E there is no unique mathematical model specified by (3.7) and, furthermore, we show by simulation that different models yield different practical results. Thus the non-uniqueness is demonstrated to be non-trivial.

To solve (3.4) for $\boldsymbol{\phi}$ with Gaussian P_E, (3.7), it is sufficient to consider a quadratic form in the velocity components u_i,

$$\phi_i = \Gamma_i + \gamma_{ijk} u_j u_k,$$

for some unknown tensors $\boldsymbol{\Gamma}$ and $\boldsymbol{\gamma}$. By equating coefficients of the u_i in (3.4), it follows that

$$-\tfrac{1}{2}\frac{\partial \lambda_{kl}}{\partial x_j} u_j u_k u_l = \lambda_{ij}\gamma_{ilm} u_j u_l u_m \tag{4.1a}$$

and

$$-\tfrac{1}{2}\lambda^{-1}\frac{\partial \lambda}{\partial x_j} u_j = -\gamma_{ijk}(\delta_{ij} u_k + \delta_{ik} u_j) + \Gamma_i \lambda_{ij} u_j \sigma^{-2}. \tag{4.1b}$$

A simple solution of (4.1*a*) is

$$\gamma_{ijk} = -\tfrac{1}{2}\lambda_{il}^{-1}\frac{\partial \lambda_{jk}}{\partial x_l}. \tag{4.2a}$$

However, because of the summation in (4.1*a*) it is possible to permute the indices cyclicly and find the alternative solutions

$$\gamma_{ijk} = -\tfrac{1}{2}\lambda_{il}^{-1}\frac{\partial \lambda_{ik}}{\partial x_j} = \tfrac{1}{2}\lambda_{kl}\frac{\partial \lambda_{il}^{-1}}{\partial x_j} \tag{4.2b}$$

and

$$\gamma_{ijk} = -\tfrac{1}{2}\lambda_{il}^{-1}\frac{\partial \lambda_{lj}}{\partial x_k} = \tfrac{1}{2}\lambda_{jl}\frac{\partial \lambda_{il}^{-1}}{\partial x_k}. \tag{4.2c}$$

Actually, any linear combination of the three solutions given above, such that the weights sum to one, is also a solution.

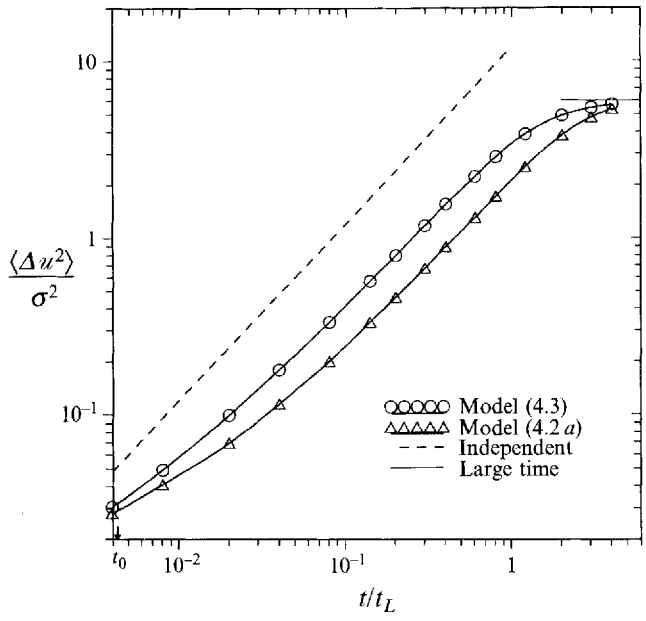

FIGURE 1. Mean-square relative velocity increment, $\langle \Delta u^2 \rangle = \langle (u_i^{(1)} - u_i^{(2)})^2 \rangle$ where $u^{(1)} = u^{(1)} - u_0^{(1)}$ etc. is shown for two different stochastic models. Independent particle motion is shown as the dashed line. Parameters are $C_0 = 4$, $\epsilon = 1$, $\sigma = 1$, $\Delta_0 = 10^{-4}$ ($t_0 = 10^{-8/3}$), and 2×10^4 particle-pair trajectories are simulated.

From (4.1*b*) it follows that for the first solution (4.2*a*)

$$\Gamma_i = \sigma^2 \lambda_{ij}^{-1} \lambda^{-1} \frac{\partial \lambda}{\partial x_j},$$

while Γ_i vanishes for both of the latter two solutions. Notice that these last results are equivalent to the average Eulerian acceleration vanishing, i.e. $\langle \boldsymbol{a} \rangle = \boldsymbol{o}$, which requires

$$\Gamma_i + \sigma^2 \gamma_{ijk} \lambda_{jk}^{-1} = 0,$$

but has been arrived at without explicitly invoking that constraint.

Notice also that the first solution (4.2*a*) is symmetric in the indices j and k while the latter two (4.2*b*, *c*) are not. Because the tensor γ_{ijk} appears in $\boldsymbol{a}$ as the combination $\gamma_{ijk} u_j u_k$, only that part of γ_{ijk} which is symmetric with respect to j and k is significant. This symmetry property will prove useful in §5 as it avoids some redundancy. Thus solutions (4.2*b*) and (4.2*c*) are essentially equivalent because the difference between them is antisymmetric with respect to the indices j and k. In fact, the solution obtained from the normalized sum,

$$\gamma_{ijk} = \tfrac{1}{4} \lambda_{kl} \frac{\partial \lambda_{il}^{-1}}{\partial x_j} + \tfrac{1}{4} \lambda_{jl} \frac{\partial \lambda_{il}^{-1}}{\partial x_k}, \tag{4.3}$$

which is symmetric in j and k, is in some sense more fundamental. Solution (4.2*c*), or equivalently (4.3), was presented first by Thomson (1990).

Without explicit consideration of the non-unique part of $\boldsymbol{a}$, the solution we have sought has in any case illustrated the problem. Figure 1 shows results of numerical calculations with (3.5) for the mean-square relative velocity increments, $\langle (u_i^{(1)} - u_i^{(2)})^2 \rangle$ where $u^{(1)} = u^{(1)} - u_0^{(1)}$ etc. The calculations were performed with $C_0 = 4$, and an initial separation of $\Delta_0 = 10^{-4} L$, using two models, (4.3) and (4.2*a*). The smallness of Δ_0

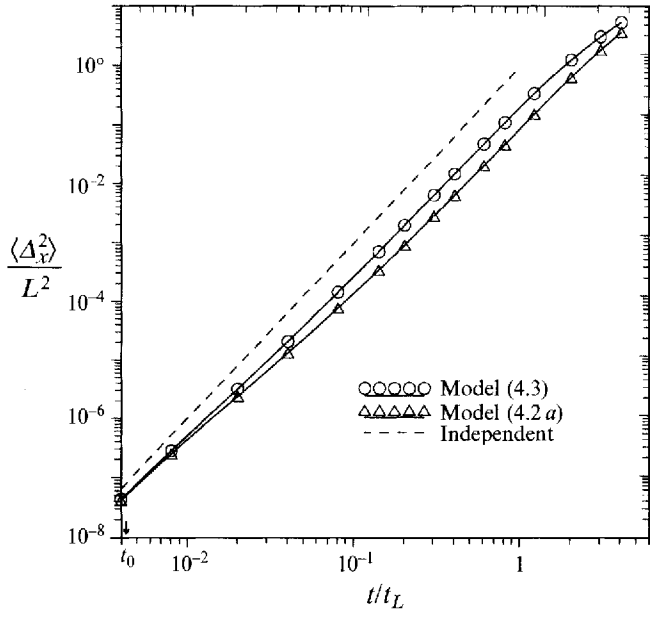

FIGURE 2. Mean-square relative displacement, $\langle \Delta_x^2 \rangle = \langle (x_i^{(1)} - x_i^{(2)})^2 \rangle$ where $x^{(1)} = \boldsymbol{x}^{(1)} - \boldsymbol{x}_0^{(1)} - \boldsymbol{u}_0^{(1)} t$ etc., is shown for parameters corresponding to figure 1.

ensures that the inertial range of turbulence is resolved (Borgas & Sawford 1991). These results show that the alternative models produce significantly different dispersion statistics. Both sets of results agree for very short times and very large times and differ most in the inertial sub-range. The inertial range for both models has the correct linear power-law dependence

$$\langle (u_i^{(2)} - u_i^{(1)})^2 \rangle = \mathscr{C} \epsilon t + O(t^2) \quad \text{for} \quad t_0 \ll t \ll t_L, \tag{4.4}$$

but the constant of proportionality, $\mathscr{C}$, is model dependent. The inner timescale, $t_0 = (\Delta_0^2/\epsilon)^{1/3}$, is much smaller than t_L in all of our work. The stochastic models give different values for $\mathscr{C}$ because of the different acceleration structure corresponding to different $\boldsymbol{a}$-vectors. Moreover, these models differ from that obtained by assuming that particle-pair cross-accelerations are negligible (Novikov 1963), since then $\mathscr{C} = 6C_0$. Details of the acceleration structure are given later.

Mean-square separation increments, which are of considerable practical concern because of their relation to concentration fluctuations and the dispersion of clouds of material, are also model dependent, as shown in figure 2. Once again the inertial sub-range is of primary concern with

$$\langle (\Delta_i - \Delta_{0i})^2 \rangle = \tilde{\mathscr{C}} \epsilon t^3 + O(t^4) \quad \text{for} \quad t_0 \ll t \ll t_L \tag{4.5}$$

having a model-dependent coefficient $\tilde{\mathscr{C}}$, which in general differs from the negligible-cross-acceleration case, $\tilde{\mathscr{C}} = 2C_0$.

There are no exact results for the values of the inertial sub-range constants $\mathscr{C}$ and $\tilde{\mathscr{C}}$ and no reliable measurements. All estimates involve (sometimes obscure) closure assumptions. For example, Fung *et al.* (1992) use a kinematic simulation of randomly forced Fourier modes to represent a velocity field and find quite small values for $\mathscr{C}$ and $\tilde{\mathscr{C}}$. Because these estimates cannot be regarded as definitive, we treat the values of these constants as unknown and hence cannot discriminate between various models on this basis.

5. Non-uniqueness: a family of models

In this section we further extend the quadratic-form class of models corresponding to Gaussian P_E. Only some of the quadratic-form solutions of (3.4) have so far been considered. It is useful to construct a general form of such quadratic solutions, of which (4.2a) and (4.3) are special cases. This is a convenient family to consider and later we shall see that there are dynamical reasons for the choice. For the present, we get an indication of what general procedures are required by examining this fairly broad general subset of all possible forms for $\boldsymbol{a}$ in an objective way.

Setting $\boldsymbol{\phi} = \hat{\boldsymbol{\phi}} + \tilde{\boldsymbol{\phi}}$, where $\hat{\boldsymbol{\phi}}$ corresponds to the quadratic form (4.2a) which we adopt as a 'reference' model, and

$$\tilde{\phi}_i = \tilde{\Gamma}_i + \tilde{\gamma}_{ijk} u_j u_k, \tag{5.1}$$

the tensor coefficients, which depend upon Δ, are sought such that

$$\frac{\partial}{\partial u_i}(P_E \tilde{\phi}_i) = 0.$$

Linear terms in the velocity are omitted because they are found to be trivial (i.e. vanish identically). Our aim therefore is to find the most general form of the tensor $\tilde{\gamma}_{ijk}$; it transpires that the structure is determined by just five arbitrary functions of Δ. For the appropriate choice of these functions we can recover (4.3) from the general family. The details are given in the remainder of the section, and while these details are essential for the technical calculations, it is not necessary to understand them to proceed with the remainder of the work.

Using $\partial P_E/\partial u_i = -\lambda_{ij} u_j P_E/\sigma^2$, the 'solenoidal' property is equivalent to two independent equations:

$$\lambda_{ij} \tilde{\gamma}_{ilm} u_j u_l u_m = 0 \tag{5.2}$$

and

$$-\tilde{\gamma}_{ijk}(\delta_{ij} u_k + \delta_{ik} u_j) + \tilde{\Gamma}_i \lambda_{ij} u_j \sigma^{-2} = 0. \tag{5.3}$$

Now let $\lambda_{ij} \tilde{\gamma}_{ilm} = \Omega_{jlm}$, whereupon (5.2) gives

$$\Omega_{ijk} + \Omega_{kij} + \Omega_{jki} = 0 \quad \forall\, i,j \quad \text{and} \quad k. \tag{5.4}$$

Here we have already used symmetry in the second two indices of Ω_{ijk}, j and k. At this point it is necessary to consider the 'block' structure of Ω_{ijk} using again the notation of (3.1). It is also useful to corrupt some of the former notation such that superscript indices in braces simply differentiate between the eight sub-tensors which constitute Ω_{ijk}:

$$\Omega^{\{1\}}_{i'j'k'};\quad \Omega^{\{2\}}_{i''j'k'};\quad \Omega^{\{3\}}_{i'j''k'};\quad \Omega^{\{4\}}_{i''j''k'};\quad \Omega^{\{5\}}_{i'j'k''};\quad \Omega^{\{6\}}_{i''j'k''};\quad \Omega^{\{7\}}_{i'j''k''};\quad \Omega^{\{8\}}_{i''j''k''}, \tag{5.5}$$

where the range of the indices is prescribed in (3.1) and, for example, the fourth tensor listed has $i > 3, j > 3$ but $k \leqslant 3$. The sub-tensors are not all independent and symmetry properties must be satisfied. For example, particle-labelling symmetry requires

$$\Omega^{\{1\}}_{i'j'k'} = -\Omega^{\{8\}}_{i'j'k'};\quad \Omega^{\{2\}}_{i''j'k'} = -\Omega^{\{7\}}_{i'j'k'};$$
$$\Omega^{\{3\}}_{i'j''k'} = -\Omega^{\{6\}}_{i'j'k'} \quad \text{and} \quad \Omega^{\{4\}}_{i''j''k'} = -\Omega^{\{5\}}_{i'j''k'}.$$

The minus signs arise because each of the sub-tensors is an odd function of Δ which follows from isotropy and $\Delta \to -\Delta$ upon labelling reversal (cf. (3.8)).

Furthermore, from the symmetry

$$\Omega_{ijk} = \Omega_{ikj},$$

it follows that all the tensors except $\Omega^{\{3\}}_{i'j'k'}$ and $\Omega^{\{4\}}_{i'j'k'}$, are likewise symmetric in the latter two indices, while for those exceptions

$$\Omega^{\{3\}}_{i'j'k'} = -\Omega^{\{4\}}_{i'k'j'}.$$

Thus the symmetries greatly reduce the number of unknown quantities. In addition, substituting the sub-tensors into (5.4), results in eight equations, of which two are independent:

$$\Omega^{\{1\}}_{i'j'k'} + \Omega^{\{1\}}_{k'i'j'} + \Omega^{\{1\}}_{j'k'i'} = 0 \quad \forall\, i', j' \quad \text{and} \quad k' \tag{5.6a}$$

and

$$\Omega^{\{2\}}_{i'j'k'} + \Omega^{\{3\}}_{k'i'j'} - \Omega^{\{4\}}_{j'k'i'} = 0 \quad \forall\, i', j' \quad \text{and} \quad k'. \tag{5.6b}$$

Finally, isotropy implies that

$$\Omega^{\{i\}}_{i'j'k'} = (\tilde{A}^{\{i\}} - \tilde{B}^{\{i\}} - \tilde{C}^{\{i\}} - \tilde{D}^{\{i\}})\frac{\Delta_{i'}\Delta_{j'}\Delta_{k'}}{\Delta^3} + \tilde{B}^{\{i\}}\frac{\Delta_{i'}}{\Delta}\delta_{j'k'} + \tilde{C}^{\{i\}}\frac{\Delta_{j'}}{\Delta}\delta_{i'k'} + \tilde{D}^{\{i\}}\frac{\Delta_{k'}}{\Delta}\delta_{i'j'} \tag{5.7}$$

where the coefficients are functions of Δ. For those tensors which are symmetric, i.e. $i = 1, 2$, we have $\tilde{C}^{\{i\}} = \tilde{D}^{\{i\}}$. In conjunction with (5.6*a*) and (5.6*b*), (5.7) implies that

$$\tilde{A}^{\{1\}} = 0; \quad \tilde{B}^{\{1\}} + 2\tilde{\mathscr{C}}^{\{1\}} = 0; \quad \tilde{B}^{\{2\}} + 2\tilde{C}^{\{3\}} = 0; \quad \tilde{A}^{\{2\}} + 2\tilde{A}^{\{3\}} = 0; \quad \tilde{B}^{\{3\}} + \tilde{C}^{\{2\}} + \tilde{D}^{\{3\}} = 0. \tag{5.8}$$

Therefore, the arbitrariness of the quadratic-form family reduces to five unknown functions of Δ: $\tilde{A}^{\{2\}}$, $\tilde{B}^{\{1\}}$, $\tilde{B}^{\{2\}}$, $\tilde{C}^{\{2\}}$ and $\tilde{D}^{\{3\}}$.

Hence further non-trivial quadratic-form modifications to the solutions we have already obtained are possible. It can be shown that when considered in the form $\tilde{\gamma}_{ijk}u_j u_k$ (which is how the terms contribute to $\boldsymbol{a}$) the new terms are a linear combination of vectors in the directions $\boldsymbol{u}^{(1)}$, $\boldsymbol{u}^{(2)}$ and $\boldsymbol{\Delta}$, and the coefficients are functions of the scalars $u_i^{(1)}u_i^{(1)}$, $u_i^{(2)}u_i^{(2)}$, $u_i^{(1)}u_i^{(2)}$, Δ, $\Delta_i u_i^{(1)}$ and $\Delta_i u_i^{(2)}$, exactly as required by isotropy (cf. (3.6)). Solutions (4.2) and (4.3) are, of course, also consistent with isotropy.

By definition, the reference model (4.2*a*) is a member of the quadratic-form family of models: given by $\tilde{A}^{\{2\}}$, $\tilde{B}^{\{1\}}$, $\tilde{B}^{\{2\}}$, $\tilde{C}^{\{2\}}$ and $\tilde{D}^{\{3\}}$ all zero. Likewise, Thomson's model (4.3) can also be represented by (5.1) with

$$\left.\begin{aligned} \tilde{A}^{\{2\}} &= -\frac{1}{2}\frac{f'}{(1+f)^2}, \quad \tilde{B}^{\{1\}} = -\frac{gg'}{(1-g^2)^2} - \frac{1}{4}\frac{(f+g)f'}{(1-f^2)(1-g^2)}, \\ \tilde{B}^{\{2\}} &= \frac{gg'}{(1-g^2)^2} + \frac{1}{4}\frac{(1+fg)f'}{(1-f^2)(1-g^2)}, \quad \tilde{D}^{\{3\}} = -\frac{1}{4}\frac{(1+g^2)g'}{(1-g^2)^2} + \frac{1}{8}\frac{(f+g)-2(1+fg)}{(1-f^2)(1-g^2)}f', \\ \tilde{C}^{\{2\}} &= -\frac{1}{4}\frac{(1+g^2)g'}{(1-g^2)^2} - \frac{1}{8}\frac{(f+g)f''}{(1-f^2)(1-g^2)} + \frac{1}{8}\frac{f'}{(1+f)(1+g)}. \end{aligned}\right\} \tag{5.9}$$

Note that, because the Eulerian constraint (3.4) is linear, the family of solutions $\boldsymbol{\phi}$ (the full solution $\boldsymbol{\phi} = \hat{\boldsymbol{\phi}} + \tilde{\boldsymbol{\phi}}$) is closed under the addition operation.

6. Two-to-one reduction

Given that the stochastic models are non-unique according to (3.4), it is necessary to consider further criteria for their selection and specification. That is the purpose of this section, and we restore generality to P_E for the discussion. Consequently, the models are also allowed to be more general than the quadratic-form family considered earlier.

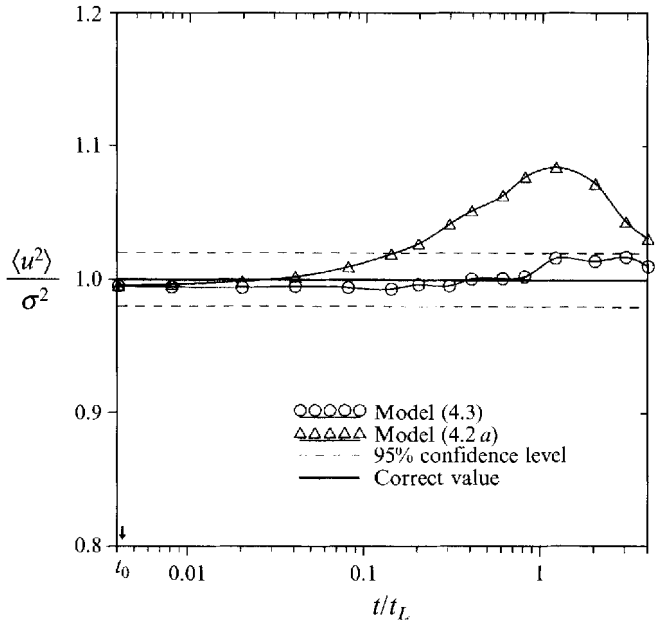

FIGURE 3. The velocity-component variance, $\langle u^2\rangle = \frac{1}{6}\langle u_i u_i\rangle$, is shown for the simulation corresponding to figure 1. The expected unit value for stationary turbulence is modelled satisfactorily well by model (4.3), with confidence levels (95%) shown as the dashed lines.

A useful test of two-particle models is their ability to generate appropriate one-particle statistics. For example, suppose that we have a large ensemble of particle-pair trajectories; then by selecting from each pair the trajectory corresponding to particle one we can construct an ensemble of one-particle trajectories. The statistics of the latter ensemble must agree with those generated by an explicit one-particle model, say (2.8). Note that this principle is not connected with the coincidence principle which says that if the particle separation, A, vanishes then the particles must move as one. Because our models are constructed on the basis of separation greater than η, no matter how small η (i.e. how large Re) the particles will separate according to inertial-range velocities and will always move apart significantly in a finite time.

Thomson (1990) recognized the importance of two-to-one reduction but did not address the issue in great depth; here, however, we derive useful constraints according to that principle. Thomson (1990) chose to examine in detail only the properties of the mean-square velocity fluctuation. Figure 3 compares the one-particle velocity-component variance calculated from the two models (4.2a) and (4.3) with the exact one-particle velocity-component variance, a constant value σ^2. It can be seen that the model corresponding to (4.2a) shows a systematic deviation from the correct value, whereas the model corresponding to (4.3) has deviations which are of the order of the expected errors due to computer simulation (bounded by the dashed lines). Here, the standard error is approximately $\pm\sigma^2(2/N)^{\frac{1}{2}}$ where N is 20000; then $\pm 0.02\sigma^2$ is a 95% confidence level. Similar results apply to the respective one-particle displacements (figure 4). Again the two-to-one results obtained numerically for model (4.3) are rather good and the results for model (4.2a) less so. The exact result in figure 4 follows from (2.9). Thus these results indicate that some models perform better than others, as judged by one-particle statistics.

The two-to-one reduction principle can be applied to many different statistics, the Lagrangian velocity variance being simply one example. In this section, however, the

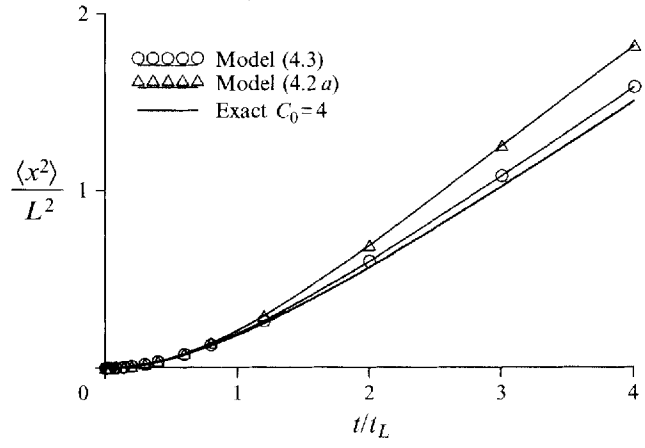

FIGURE 4. One-particle mean-square single-component displacement, $\langle x^2 \rangle = \frac{1}{6}\langle (x_i - x_{0i})^2 \rangle$, is shown for the stochastic simulations of figure 1, and compared with the exact one-particle solution from (2.9).

main emphasis is on two Eulerian constraints: (6.8), which says that the 'acceleration' of particle one, conditioned on the velocity of particle one at some instant, is the same regardless of whether it is calculated from a one- or two-particle model; and (6.10), which is concerned with the acceleration conditioned on the velocity of a second particle a fixed distance away. The latter constraint is shown to depend explicitly upon the Navier–Stokes equations.

The rationale for stressing these constraints over any others, and in particular over the velocity variance, is twofold. Firstly, only Eulerian constraints lead to manageable mathematical analysis, and, secondly, the 'accelerations' arise naturally at the lowest non-trivial order in a small-time expansion of Lagrangian behaviour.

The remainder of this section furnishes the details and may be studied selectively. The essential results are (6.8) and (6.10) although the derivation of (6.12) from (6.10) with the accompanying use of the Navier–Stokes equations is also crucial. It must be stressed that these results are for general P_E and also for fully general forms of the stochastic models.

Consider the general two-particle probability distribution

$$P_{\mathscr{S}}(\boldsymbol{u}, \boldsymbol{x}, t) = \mathscr{I}^{-2} \int P_L(\boldsymbol{u}, \boldsymbol{x}, t; \boldsymbol{u}_0, \boldsymbol{x}_0, t_0)\, P_E(\boldsymbol{u}_0; \boldsymbol{x}_0, t_0)\, \mathscr{S}(\boldsymbol{x}_0)\, \mathrm{d}^6 \boldsymbol{u}_0\, \mathrm{d}^6 \boldsymbol{x}_0 \tag{6.1}$$

representing the statistics of particle pairs initially distributed in space according to $\mathscr{I}^{-2}\mathscr{S}(\boldsymbol{x}) = \mathscr{I}^{-2}\tilde{\mathscr{S}}(\boldsymbol{x}^{(1)})\,\tilde{\mathscr{S}}(\boldsymbol{x}^{(2)})$. The normalization factor $\mathscr{I}$, where

$$\mathscr{I} = \int \tilde{\mathscr{S}}(\boldsymbol{x}^{(1)})\, \mathrm{d}^3 \boldsymbol{x}^{(1)},$$

requires that $\tilde{\mathscr{S}}$ is suitably integrable. Note that the Eulerian p.d.f. is recovered from the special limit where $\tilde{\mathscr{S}}$ is a uniform distribution and the Lagrangian p.d.f. corresponds to the case where $\tilde{\mathscr{S}}$ is a δ-function. However, here we assume that $\tilde{\mathscr{S}}$ is suitably smooth (with arbitrarily many derivatives existing and bounded) so that Taylor-series expansions of (6.1) are useful.

A fundamental property of $P_{\mathscr{S}}$ is that it reduces to a one-particle p.d.f. on integration over $\boldsymbol{u}^{(2)}$ and $\boldsymbol{x}^{(2)}$, i.e.

$$\int P_{\mathscr{S}}(\boldsymbol{u}, \boldsymbol{x}, t)\, \mathrm{d}^3 \boldsymbol{u}^{(2)}\, \mathrm{d}^3 \boldsymbol{x}^{(2)} = P_{\tilde{\mathscr{S}}}(\boldsymbol{u}^{(1)}, \boldsymbol{x}^{(1)}, t)\, \mathscr{I}. \tag{6.2}$$

Consequently, one-particle statistics calculated from the two-particle p.d.f. are identical with those calculated from the one-particle p.d.f.

Now (6.2) must hold for any sensible model p.d.f. It can be shown that any smooth solution to the two-particle Fokker–Planck equation is a well-defined p.d.f. in the sense that when integrated as in (6.2) it produces a function with the properties of a one-particle p.d.f. In addition, we further require that the one-particle p.d.f. be a solution of the one-particle Fokker–Planck equation (2.4) under the appropriate initial conditions. In this way we can ensure that one-particle statistics calculated from the two-particle model are identical to those calculated from the one-particle model. We refer to this requirement as the two-to-one reduction property. Clearly, since the model two-particle p.d.f. is a function of $\boldsymbol{a}$, this property constrains the form of $\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{x}, t)$.

We consider, then, a p.d.f. $P_{\mathscr{S}}$ satisfying the Fokker–Planck equation which we write as

$$\frac{\partial P_{\mathscr{S}}}{\partial t} = \mathscr{L}\{P_{\mathscr{S}}\}, \tag{6.3}$$

with the operator $\mathscr{L}$ given by

$$\mathscr{L}\{f\} = -u_j \frac{\partial f}{\partial x_j} - \frac{\partial}{\partial u_j}(a_j f) + \tfrac{1}{2} C_0 \epsilon \frac{\partial^2 f}{\partial u_j \partial u_j}, \tag{6.4}$$

with repeated indices meaning summation for $j = 1$–6. Henceforth we consider only stationary turbulence, so $P_E(\boldsymbol{u}; \boldsymbol{x}, t) = P_E(\boldsymbol{u}; \boldsymbol{x})$, with $\mathscr{L}\{P_E\} = 0$.

According to our two-to-one reduction property, $P_{\tilde{\mathscr{S}}}$ in (6.2) satisfies a Fokker–Planck equation like (6.3), but with an operator $\mathscr{L}'$ given by (6.4) for an appropriate three-dimensional one-particle $\boldsymbol{a}$-vector, i.e. (2.5) with $\phi_i = 0$ and with summation of repeated indices only from 1 to 3. Let this one-particle $\boldsymbol{a}$-vector be denoted by $\tilde{\boldsymbol{a}}(\boldsymbol{u}^{(1)})$ (for particle one say).

At the initial instant, $t = t_0$, $P_{\mathscr{S}} = \mathscr{S}^{-2} P_E(\boldsymbol{u}; \boldsymbol{\Delta}, t_0)$, and with a corresponding result for $P_{\tilde{\mathscr{S}}}$, (6.2) reduces to

$$\int P_E(\boldsymbol{u}; \boldsymbol{\Delta})\, \mathscr{S}(\boldsymbol{x})\, \mathrm{d}^3\boldsymbol{u}^{(2)}\, \mathrm{d}^3\boldsymbol{x}^{(2)} = P_E(\boldsymbol{u}^{(1)}; \boldsymbol{x}^{(1)})\, \tilde{\mathscr{S}}(\boldsymbol{x}^{(1)})\, \mathscr{S}, \tag{6.5}$$

where P_E under the integral sign, as is indicated by the arguments, refers to the two-point form. The Eulerian equation, (6.5), can be satisfied identically because we are free to construct the Eulerian probability distributions appropriately. Nevertheless, it remains to be shown that, for any particular choice of $\boldsymbol{a}$ (in the two-particle model), the Lagrangian reduction (6.2) occurs properly. This difficult problem can be tackled by expanding the probability distributions in a Taylor series about t_0 while keeping $\boldsymbol{u}$ and $\boldsymbol{x}$ fixed. The reason is that the statistics of the initial state are known and therefore we may explicitly calculate the terms that arise in the expansion. For instance, considering the two-particle statistics we have

$$P_{\mathscr{S}}(\boldsymbol{u}, \boldsymbol{x}, t) = P_E(\boldsymbol{u}; \boldsymbol{x})\, \mathscr{S}(\boldsymbol{x})\, \mathscr{S}^{-2} + (t - t_0) \frac{\partial P_{\mathscr{S}}}{\partial t}\Big|_{t=t_0} + \tfrac{1}{2}(t - t_0)^2 \frac{\partial^2 P_{\mathscr{S}}}{\partial t^2}\Big|_{t=t_0} + \ldots \tag{6.6}$$

The time derivatives in (6.6) may be evaluated from the Fokker–Planck equation in terms of P_E and $\mathscr{S}(\boldsymbol{x})$. Thus it is possible to successively calculate terms of $O((t - t_0)^n)$ for larger and larger n on the left-hand side of (6.2).The right-hand side of (6.2) can similarly be expanded in a Taylor series in $t - t_0$, which is like (6.6) except that one-particle quantities are used. Thus we have an established hierarchy of constraints by equating powers of $t - t_0$.

Consider the $O(t-t_0)$ terms. The result for the first-order time derivative is

$$\frac{\partial P}{\partial t}\mathscr{S}|_{t=t_0} = \mathscr{L}\{P_{\mathscr{S}}\}|_{t=t_0} = \mathscr{L}\{P_E\mathscr{S}\} = -u_j\frac{\partial \mathscr{S}}{\partial x_j}P_E. \tag{6.7}$$

Notice that this result depends upon $\mathscr{L}\{P_E\}$ vanishing. Substitution into the left-hand side of (6.2) leads to the term

$$\int P_E(\boldsymbol{u};\boldsymbol{x})\,u_j\frac{\partial \mathscr{S}}{\partial x_j}\,\mathrm{d}^3\boldsymbol{u}^{(2)}\,\mathrm{d}^3\boldsymbol{x}^{(2)} = \int P_E(\boldsymbol{u};\boldsymbol{\Delta})\left(u_j^{(2)}\frac{\partial \mathscr{S}}{\partial x_j^{(2)}}+u_j^{(1)}\frac{\partial \mathscr{S}}{\partial x_j^{(1)}}\right)\mathrm{d}^3\boldsymbol{u}^{(2)}\,\mathrm{d}^3\boldsymbol{x}^{(2)}$$

$$= \int P_E(\boldsymbol{u}^{(1)})\left(\langle u_j^{(2)}\,|\,\boldsymbol{u}^{(1)}\rangle\frac{\partial \mathscr{S}}{\partial x_j^{(2)}}\right)\mathrm{d}^3\boldsymbol{x}^{(2)}+u_j^{(1)}\frac{\partial \tilde{\mathscr{S}}}{\partial x_j^{(1)}}P_E(\boldsymbol{u}^{(1)}).$$

The notation $\langle\,|\,\boldsymbol{u}^{(1)}\rangle$ indicates an Eulerian ensemble average over $\boldsymbol{u}^{(2)}$ with the velocity at $\boldsymbol{x}^{(1)}$ fixed (equal to $\boldsymbol{u}^{(1)}$). Following an application of Green's theorem,† i.e.

$$\int f_j(\boldsymbol{x}^{(2)})\frac{\partial g}{\partial x_j^{(2)}}\,\mathrm{d}^3\boldsymbol{x}^{(2)}+\int g\left(\frac{\partial}{\partial x_j^{(2)}}f_j(\boldsymbol{x}^{(2)})\right)\mathrm{d}^3\boldsymbol{x}^{(2)} = \int\frac{\partial}{\partial x_j^{(2)}}(gf_j(\boldsymbol{x}^{(2)}))\,\mathrm{d}^3\boldsymbol{x}^{(2)} = 0,$$

the integral on the right-hand side vanishes by continuity (since $\langle\,|\,\rangle$ and $\partial/\partial x_j^{(2)}$ commute). The residual term,

$$-u_j^{(1)}\frac{\partial \tilde{\mathscr{S}}}{\partial x_j^{(1)}}P_E(\boldsymbol{u}^{(1)}),$$

is identical with the $O(t-t_0)$ Taylor series expansion of the right-hand side of (6.2), i.e. (6.2) is an identity at $O(t-t_0)$.

Higher orders become increasingly complex algebraically, but the principle is the same. The $O((t-t_0)^2)$ terms lead to two new (vector) constraints for $\boldsymbol{a}$:

$$\langle a_i^{(1)}\,|\,\boldsymbol{u}^{(1)}\rangle = \tilde{\boldsymbol{a}}(\boldsymbol{u}^{(1)}) = -t_L^{-1}u_i^{(1)}, \tag{6.8}$$

which is independent of $\boldsymbol{x}^{(2)}$ (and $\boldsymbol{x}^{(1)}$) and

$$\langle a_i^{(2)}\,|\,\boldsymbol{u}^{(1)}\rangle = \Psi_i+\frac{\partial}{\partial x_j^{(2)}}\langle u_j^{(2)}u_k^{(2)}\,|\,\boldsymbol{u}^{(1)}\rangle \tag{6.9}$$

for any Ψ_i such that $\partial\Psi_i/\partial x_i^{(2)} = 0$.

Equation (6.8) follows directly from the physical concept of conditioned acceleration of a particle and may be arrived at without our elaborate expansion procedure and indeed appears to be the obvious two-to-one constraint from direct inspection of the Fokker–Planck equation. On the other hand, interpretation of (6.9) is less obvious but apparently has equal importance for the correct two-to-one reduction of the probability. However, it also has a physical interpretation when we identify $\langle a_i^{(2)}\,|\,\boldsymbol{u}^{(1)}\rangle$ with the acceleration of particle two's conditioned velocity:

$$\langle a_i^{(2)}\,|\,\boldsymbol{u}^{(1)}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle u_i^{(2)}(t)\,|\,\boldsymbol{u}^{(1)}(t')\rangle|_{t=t'}.$$

The following kinematic relationship can then be derived:

$$\langle a_i^{(2)}\,|\,\boldsymbol{u}^{(1)}\rangle = \left\langle\frac{\mathrm{d}u_i^{(2)}}{\mathrm{d}t}\,|\,\boldsymbol{u}^{(1)}\right\rangle = \left\langle\frac{\partial u_i^{(2)}}{\partial t}\,|\,\boldsymbol{u}^{(1)}\right\rangle+\frac{\partial}{\partial x_j^{(2)}}\langle u_j^{(2)}u_i^{(2)}\,|\,\boldsymbol{u}^{(1)}\rangle, \tag{6.10}$$

† To apply Green's theorem we need to make some assumptions about the behaviour of the functions at 'infinity'; however, it is always possible to restrict the class of functions from which $\mathscr{S}$ is drawn so that the conditions are satisfied.

provided that the separation of points is such that $\Delta \gg \eta$ which is always true in the present context. This means that (6.10) is an inertial-range quantity when $\Delta \gg \eta$, and in particular the velocity increments similarly conditioned, i.e.

$$\langle u_i^{(2)}(t') - u_i^{(2)}(t) \,|\, \boldsymbol{u}^{(1)}(t) \rangle \quad (t' > t)$$

vary with inertial-range timescales rather than $t_\eta = (\bar{\epsilon}/\nu)^{\frac{1}{2}}$, which would be the shortest relevant timescale for one-particle statistics. Thus with

$$\Psi_i = \left\langle \frac{\partial u_i^{(2)}}{\partial t} \,|\, \boldsymbol{u}^{(1)} \right\rangle \tag{6.11}$$

(and therefore $\partial \Psi_i / \partial x_i^{(2)} = 0$) the two-to-one reduction is correct at $O((t-t_0)^2)$. Note that, since the time derivative and conditional average do not commute, stationarity does not necessarily imply that Ψ_i vanishes. This is because P_E is a two-point (therefore spatial derivatives commute), not a two-time, p.d.f. Models that satisfy (6.9) with Ψ_i different from (6.11) are not reasonable models of fluid-particle accelerations, but are acceptable statistical models purely on the two-to-one reduction basis. Since we are concerned here with models of fluid-flow turbulence we shall insist upon (6.11) and not consider any more general models.

Although the constraints (6.8) and (6.10) can be arrived at by evaluating the appropriate acceleration statistics directly, the expansion procedure set out here shows the importance of such constraints in ensuring the proper two-to-one reduction, and furthermore generates and identifies a hierarchy of constraints in a systematic way.

The Eulerian distribution, P_E, allows the computation of the spatial derivative of the conditional average, but not the time derivative. This term could be written down if we had two-point and two-time Eulerian statistics, but we assume only the former. However, the calculation of (6.11) is implicit when we use the Navier–Stokes equations (conditionally averaged) to substitute for the right-hand side of (6.10) so that

$$\langle a_i^{(2)} \,|\, \boldsymbol{u}^{(1)} \rangle = -\frac{1}{\rho} \left\langle \frac{\partial p^{(2)}}{\partial \Delta_i} \,|\, \boldsymbol{u}^{(1)} \right\rangle + \nu \left\langle \frac{\partial^2 u_i^{(2)}}{\partial \Delta_j \partial \Delta_j} \,|\, \boldsymbol{u}^{(1)} \right\rangle + \langle f_i^{(2)} \,|\, \boldsymbol{u}^{(1)} \rangle \tag{6.12}$$

where all the terms on the right-hand side can be determined purely from the knowledge of P_E and prescribed external forcing statistics (properties of $\boldsymbol{f}$). For example, the viscous-stress term may be evaluated for our Gaussian p.d.f. as

$$\nu \left\langle \frac{\partial^2 u_i^{(2)}}{\partial \Delta_j \partial \Delta_j} \,|\, \boldsymbol{u}^{(1)} \right\rangle = \nu \frac{\partial^2 \rho_{ik}}{\partial \Delta_j \partial \Delta_j} u_k^{(1)},$$

which vanishes as $\nu \to 0$ ($Re \to \infty$) for fixed Δ. Thus the principle contribution to the right-hand side of (6.12) comes from the pressure gradient and external forcing term. These terms will be considered in more detail later.

Similarly to the $O(t-t_0)$ and $O((t-t_0)^2$ terms, more constraints arise at $O((t-t_0)^3)$, although the analysis is rather lengthy. We prefer to concentrate on those constraints already listed in this paper. However, it is appropriate here to briefly reconsider the numerical results given in figures 3 and 4 for the one-particle statistics generated by models (4.2a) and (4.3). The failure to reproduce the correct one-particle velocity variance in particular is a consequence of the general two-to-one reduction violation. For small times, Thomson (1990) shows that model (4.3) behaves like

$$\langle u_i u_i \rangle = 6\sigma^2 - \sigma^4 (\Delta^2 F(\Delta)(2\Delta F'(\Delta) + 5F(\Delta)))_{\Delta=\Delta_0} (t-t_0)^2 + O((t-t_0)^3), \tag{6.13}$$

where $F = \Delta^{-2}(f-g)$ and the coefficient of the $(t-t_0)^2$ term is not identically zero.

Clearly, model (4.3) then violates two-to-one reduction. In general, (6.13) may be written as

$$\langle u_i u_i \rangle = 6\sigma^2 + \left\langle a_i a_i + u_i u_j \frac{\partial a_i}{\partial x_j} + u_i a_j \frac{\partial a_i}{\partial u_j} + \tfrac{1}{2} C_0 \epsilon \left(2 \frac{\partial a_j}{\partial u_j} + u_i \frac{\partial^2 a_i}{\partial u_j \partial u_j} \right) \right\rangle_{\Delta = \Delta_0} (t - t_0)^2$$

(plus higher-order terms) so that setting the coefficient equal to zero may be regarded as a constraint on $\boldsymbol{a}$. Manipulation of this expansion using the fact that P_E satisfies the two-particle Fokker–Planck equation (cf. (2.4)) gives the simpler expression

$$\langle u_i u_i \rangle = 6\sigma^2 + 2 \left(\frac{\partial}{\partial x_k} \langle u_i u_k a_i \rangle \right)_{\Delta = \Delta_0} (t - t_0)^2 + \ldots, \tag{6.14}$$

which is linear in the unknown vector $\boldsymbol{a}$. In general, the models violate the reduction at second order in time increment. However, when principles (6.8) and (6.12) are satisfied, it is shown in that Appendix B that the reduction of one-particle velocity variance occurs correctly in the special case that P_E is Gaussian and for quadratic-form $\boldsymbol{a}$. Thus, principles (6.8) and (6.12) are the fundamental two-to-one constraints at second order in time increment.

7. Implementation of two-to-one constraints

In this section, we explore the possibility that the general quadratic solution derived in §5 may be constrained by, and satisfy, the two-to-one constraints (6.8) and (6.12). This of course specializes the results to Gaussian P_E, and gives the main results that: first, (6.8) provides one linear algebraic equation reducing the five unknown functions of §3 to four; while, secondly, (6.12) furnishes a further three relations finally giving a family depending on only one function of Δ. Once again the technical details can be ignored at a first reading. The critical result is (7.6), which is the third-order tensor in the quadratic-form model and which is fully determined except for the function $\varphi(\Delta)$ which is shown explicitly. Calculations are then undertaken with this reduced class of models with an emphasis on inertial-range properties. It is shown that a new model exists which is as good as (4.3) for predicting constant σ^2, but has additional theoretical underpinning. Remarkably, the relative dispersion results differ modestly between (4.3) and the new model which suggests that the behaviour of (4.3) is robust and that the new constraint (6.12) (leading to (7.5) and thence to (7.6)) is of limited practical importance. On the other hand, if we ignore the behaviour of σ^2, then very large variations of relative dispersion can occur even within the class of highly constrained quadratic-form models.

The standard technical procedure for examining two-to-one reductions, given that P_E is Gaussian and the $\boldsymbol{a}$ vector is a quadratic form in the variable $\boldsymbol{u}$, is to make the substitution

$$u_{i'}^{(2)} = \rho_{i'j'} u_{j'}^{(1)} + \vartheta_{i'j'} \xi_{j'}, \tag{7.1}$$

which defines a new dummy integration variable in the averages over $\boldsymbol{u}^{(2)}$, $\boldsymbol{\xi}$ (with dimensions of velocity). The tensor $\vartheta_{i'j'}$, which depends upon Δ, is prescribed by

$$\vartheta_{i'k'} \vartheta_{k'j'} = \delta_{i'j'} - \rho_{i'k'} \rho_{k'j'};$$

and the first two-to-one constraint (6.8) takes the form

$$\int a_{i'}^{(1)}(\boldsymbol{u}^{(1)}, \boldsymbol{\rho u}^{(1)} + \boldsymbol{\vartheta \xi}, \Delta) \exp\left(-\frac{1}{2} \left(\frac{\xi}{\sigma} \right)^2 \right) \mathrm{d}^3 \xi = -t_L^{-1} u_{i'}^{(1)} \quad \forall\, i'.$$

It is difficult to make any analytical progress for anything but the simple quadratic-form case for which the relevant results are

$$\Gamma_{i'} = \sigma^2 \gamma^{\{2\}}_{i'j'k'}(\delta_{j'k'} - \rho_{j'l'}\rho_{l'k'}), \quad \forall i', j' \text{ and } k'. \tag{7.2a}$$

$$\gamma^{\{1\}}_{i'j'k'} + \gamma^{\{3\}}_{i'l'k'}\rho_{l'j'} - \gamma^{\{4\}}_{i'j'l'}\rho_{l'k'} - \gamma^{\{2\}}_{i'l'm'}\rho_{l'j'}\rho_{m'k'} = 0, \tag{7.2b}$$

It can be shown by using either the solenoidal property (5.3) or the zero-mean acceleration constraint when averaged over $\boldsymbol{u}^{(1)}$,

$$\Gamma_i = -\sigma^2 \gamma_{ijk}\lambda^{-1}_{jk},$$

that (7.2*a*) is a consequence of (7.2*b*). In general, (7.2*b*) corresponds to three scalar equations:

$$A^{\{1\}} + 2fA^{\{3\}} - f^2 A^{\{2\}} = 0;$$

$$B^{\{1\}} + 2gB^{\{3\}} - g^2 B^{\{2\}} = 0;$$

and

$$C^{\{1\}} + fC^{\{3\}} + gD^{\{3\}} - fgC^{\{2\}} = 0,$$

where the functions involve the modifications, $\tilde{A}^{\{2\}}$, $\tilde{B}^{\{1\}}$, $\tilde{B}^{\{2\}}$, $\tilde{C}^{\{2\}}$ and $\tilde{D}^{\{3\}}$, according to the definition (5.7) of $\boldsymbol{\Omega}$. These equations apply generally to quadratic-form solutions, including both (4.2*a*) and (4.3). However, from the symmetry constraints (5.8), they are not all independent and a single equation governs this two-to-one reduction:

$$(1-fg^2)\tilde{B}^{\{1\}} - 2g(1-f)(\tilde{C}^{\{2\}} + \tilde{D}^{\{3\}}) + (f-g^2)\tilde{B}^{\{2\}} = 0. \tag{7.3}$$

Thus the two-to-one reduction for $\boldsymbol{a}$ is non-trivial since it constrains the possible quadratic-form modifications; of the five arbitrary functions identified above only four are 'independent'. Note that both solutions (4.2*a*) and (4.3) satisfy (7.3).

We now consider the second constraint, (6.12). By the same procedures as above we find that (6.12) requires

$$\gamma^{\{2\}}_{i'j'k'} + \gamma^{\{4\}}_{i'l'k'}\rho_{l'j'} - \gamma^{\{3\}}_{i'j'l'}\rho_{l'k'} - \gamma^{\{1\}}_{i'l'm'}\rho_{l'j'}\rho_{m'k'} = -\frac{1}{\rho}\frac{\partial}{\partial \Delta_{i'}}\mathscr{P}_{j'k'}(\boldsymbol{\Delta}) + f_{i'j'k'}(\boldsymbol{\Delta}) \tag{7.4}$$

where $\mathscr{P}_{j'k'}(\boldsymbol{\Delta})$ is related to the conditional pressure expectation which is determined solely from the Navier–Stokes equations and P_E, at least in the inertial range where we neglect the forcing (represented here by $f_{i'j'k'}(\boldsymbol{\Delta})$). This result, derived in Appendix A for the Gaussian form (3.7) for P_E, is

$$\langle p^{(2)} \,|\, \boldsymbol{u}^{(1)} \rangle = \mathscr{P}_{jk}(\boldsymbol{\Delta})\, u^{(1)}_j u^{(1)}_k + \sigma^2 \not{p}_3(\Delta).$$

Isotropy of the turbulence further requires that

$$\mathscr{P}_{jk}(\boldsymbol{\Delta}) = (\not{p}_1 - \not{p}_2)\frac{\Delta_j \Delta_k}{\Delta^2} + \not{p}_2 \delta_{jk},$$

where the functions $\not{p}_1$, $\not{p}_2$ and $\not{p}_3$ are listed in Appendix A.

The basis for the quadratic-form assumption can now be seen to depend essentially upon the Gaussian form for P_E, which implies through the pressure-gradient term that the conditional acceleration, $\langle \boldsymbol{a}^{(2)} \,|\, \boldsymbol{u}^{(1)} \rangle$, is a quadratic form with respect to the $\boldsymbol{u}^{(1)}$ velocity variable:

$$\langle a^{(2)}_i \,|\, \boldsymbol{u}^{(1)} \rangle = \Lambda_{ijk}\, u^{(1)}_j u^{(1)}_k + \sigma^2 \not{p}'_3 \frac{\Delta_i}{\Delta},$$

where

$$\Lambda_{ijk} = \left(\not{p}'_1 - \not{p}'_2 - \frac{2}{\Delta}(\not{p}_1 - \not{p}_2)\right)\frac{\Delta_i \Delta_j \Delta_k}{\Delta^3} + \not{p}'_2 \frac{\Delta_i}{\Delta}\delta_{jk} + \frac{1}{\Delta}(\not{p}_1 - \not{p}_2)\left(\frac{\Delta_j}{\Delta}\delta_{ik} + \frac{\Delta_k}{\Delta}\delta_{ij}\right).$$

Similarly, using the kinematic condition (6.10), but ignoring the time derivative, also yields quadratic forms in $\boldsymbol{u}^{(1)}$ for $\langle \boldsymbol{a}^{(2)} | \boldsymbol{u}^{(1)} \rangle$. It therefore seems reasonable to examine quadratic forms for $\langle \partial \boldsymbol{u}^{(2)}/\partial t | \boldsymbol{u}^{(1)} \rangle$ as well. This result is some way short of proving the absolute necessity of a quadratic form in the full velocity $\boldsymbol{u}$ when the Eulerian p.d.f. is Gaussian, but it supports the examination of quadratic forms as an important family of models. Novikov's (1963) model is linear in $\boldsymbol{u}$ and hence is consistent with the quadratic form for $\langle a_i^{(2)} | \boldsymbol{u}^{(1)} \rangle$ only for the trivial and unrealistic case that $\not{p}_1 = \not{p}_2 = \not{p}_3 = 0$, i.e. when the spatial structure of the Eulerian field is ignored.

The constraint (7.4), apart from the novel right-hand side, bears a remarkable similarity to the corresponding two-to-one reduction principle (7.2*b*). However, the new equation represents three additional constraints on the formulation of the stochastic models because the left-hand side is characterized by three scalar functions for the *jk*-symmetric isotropic form of A_{ijk}. Thus (7.4) is equivalent to the scalar equations

$$\left.\begin{aligned} (1-f^2)(1+f)\,\tilde{A}^{\{2\}} &= -\frac{1}{\rho}\not{p}_1', \\ (1-g^2)(1+f)(\tilde{B}^{\{1\}}+\tilde{B}^{\{2\}}) &= -\frac{1}{\rho}\not{p}_2', \\ -\tfrac{1}{2}(1-f)(\tilde{B}^{\{1\}}-\tilde{B}^{\{2\}})+(1-fg^2)\,\tilde{C}^{\{2\}}-(f-g^2)\,\tilde{D}^{\{3\}} &= -\frac{1}{\rho}\frac{\not{p}_1-\not{p}_2}{\Delta}, \end{aligned}\right\} \tag{7.5}$$

where the functions with tildes are the modifications (cf. (5.1)) to the reference solution (4.2*a*) in §4. Thus, unlike system (7.2*b*), (7.4) does not collapse into a single equation under the imposition of (5.8). However, because the general quadratic-form solution for $\boldsymbol{a}$ depends on four arbitrary functions since, using (7.3) we can reduce the five unknowns of §5 to four, it should, in principle, be possible to satisfy (7.5) (and hence (6.12)). Note that terms from (4.2*a*) do not contribute to (7.5) because $\langle \boldsymbol{a}^{(2)} | \boldsymbol{u}^{(1)} \rangle$ vanishes identically for that model.

Briefly, we consider two of those models that have so far been examined: the reference model with γ_{ijk} given by (4.2*a*) and Thomson's model (4.3). Both of these implicitly satisfy (7.3) so that there is no discrimination by that two-to-one reduction constraint. Figure 5 shows the calculated pressure-gradient coefficients ($\not{p}_1'$, $\not{p}_2'$ and $\not{p}_1-\not{p}_2$) in the quadratic form (A 5) for the conditionally averaged pressure gradient. The solid lines mark the terms determined (in Appendix A) from the Navier–Stokes equations. Now the reference model (4.2*a*) is defined by

$$\tilde{A}^{\{2\}} = \tilde{B}^{\{1\}} = \tilde{B}^{\{2\}} = \tilde{C}^{\{2\}} = \tilde{D}^{\{3\}} = 0.$$

Clearly this solution fails to satisfy (7.5) and in figure 5 the discrepancy is measured by the deviation of the solid lines from the Δ-axis. It is less clear, but nevertheless true, that when the modifications are given by (5.9), which defines model (4.3), system (7.5) is also violated. In figure 5, the implied pressure-gradient terms corresponding to (4.3) (the left-hand side of (7.5)) are represented by the dashed lines and thus the discrepancy is measured by the appropriate difference between the dashed and solid lines. In qualitative terms, this is at least as large as the former difference between model (4.2*a*) and the Navier–Stokes forms. Thus both models considered so far violate the two-to-one reduction.

Solving (7.3) and (7.5) for modifications that do reduce properly is straightforward. Bearing in mind that there are four unknowns and three equations, the solution is not

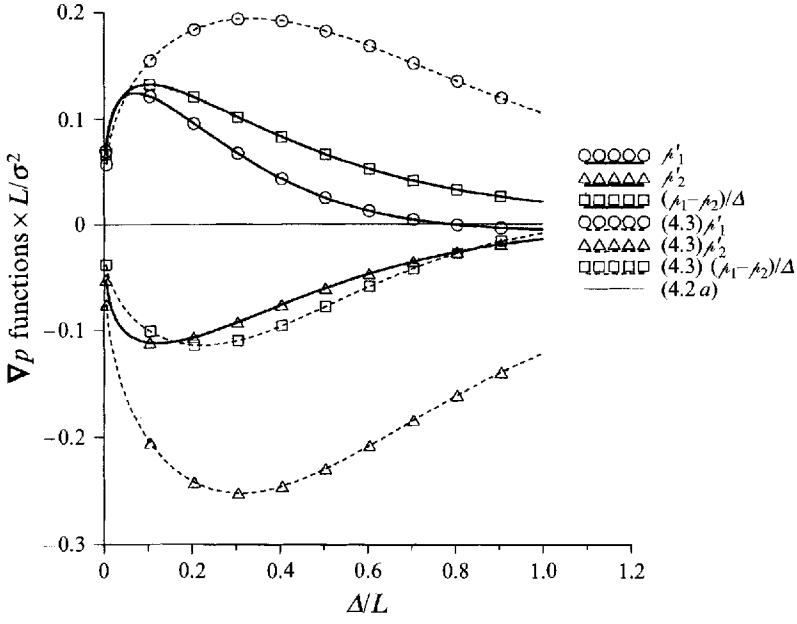

FIGURE 5. Conditional pressure-gradient functions (coefficients in the quadratic form) as determined in Appendix A and shown by the solid lines. Model (4.3) represents the corresponding pressure gradients only approximately, as shown by the dashed lines. Model (4.2a) has pressure-gradient coefficients which vanish (and lie along the Δ-axis).

unique, and the quadratic-form solution contains one arbitrary function of Δ. It is possible to represent that solution in general. Suppose we have two solutions for constrained modifications: $\tilde{\gamma}_{ijk}$ and $\tilde{\gamma}'_{ijk}$. For instance, the first can be defined to have

$$\tilde{D}^{\{3\}} = 0$$

while the second solution can be defined to have

$$\tilde{B}^{\{1\}} = \tilde{B}^{\{2\}},$$

both simply for convenience with no particular physical implications. Then a general quadratic-form solution satisfying both (7.3) and (7.5) is given by

$$\gamma_{ijk} = -\tfrac{1}{2}\lambda_{il}^{-1}\frac{\partial \lambda_{jk}}{\partial \Delta_l} + \varphi(\Delta)\tilde{\gamma}_{ijk} + (1-\varphi(\Delta))\,\tilde{\gamma}'_{ijk}, \tag{7.6}$$

where $\varphi(\Delta)$ is an arbitrary function of Δ. The inertial-range properties of Lagrangian solutions corresponding to (7.6) depend more precisely on the form of φ, so that when

$$\varphi = \varphi_0 \Delta^{-4/3} + \varphi_1 \Delta^{-2/3} + \varphi_2 + \ldots \tag{7.7}$$

the inertial-range properties are functions of the constant φ_0. If φ is more singular than (7.7) (as $\Delta \to 0$) then the stochastic-equation solutions cease to have Kolmogorov scaling properties in the inertial range. If φ is less singular than (7.7) then the inertial-range properties are those of the reference model (4.2a). In the computations examined here, we simply vary φ_0 and set $\varphi_1 = \varphi_2 = 0$. Note the latter two constants have more of a role for the larger scales ($\Delta \approx L$), rather than in the inertial range, and therefore the nature of the forcing terms, which we have neglected, may influence the choice of φ_1 and φ_2 in a more detailed study of the large scales. For the same reasons, it may be

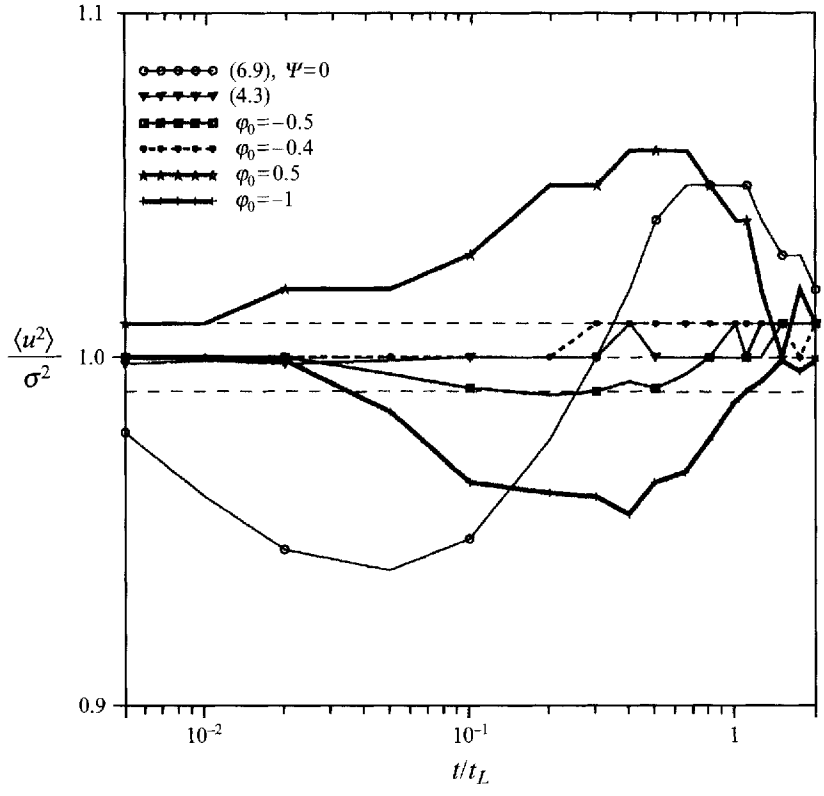

FIGURE 6. Velocity-component variance, $\langle u^2 \rangle = \frac{1}{6}\langle u_i u_i \rangle$, for the new model (7.6), for a range of φ_0 values in (7.7). The earlier simulations (figure 3) are also included. The solution with $\varphi_0 = -0.4$ is approximately correct, as is (4.3) for this statistic. The other parameters are as in figure 1.

anticipated that φ_1 and φ_2 (and the forcing) control behaviour for large times rather than during the 'inertial-range' times. We also note that the terms ignored in (7.7) are anticipated to be suitably small in the large-Δ limit, i.e. (7.7) is not strictly a power series expansion in the parameter $\Delta^{-2/3}$.

In principle, solution (7.6) is more acceptable than either the reference model (4.2*a*) or Thomson's model (4.3). However, in practice it was found that model (4.3) produced numerical simulations for the velocity-component variance (figure 3) and one-particle displacements (figure 4) which were very nearly correct. Model (7.6) can certainly do no better than this based on those criteria. Indeed, the numerical simulations for several solutions with differing values of φ_0 give a wide spectrum of velocity-variance results, which are shown in figure 6. Some solutions, despite satisfying more rigorous two-to-one constraints ((7.3) and (7.5)), have velocity fluctuations deviating by 10–12% from σ^2. Also shown on this figure are the results for a solution constrained simply by (6.9) with the Ψ_i all set to zero and which are similarly in error. However, it is apparent that by judicious choice of φ_0, the velocity-component variance is adequately modelled in the inertial range (with less than 1–2% deviations from σ^2). Likewise, these same conclusions and observations apply to one-particle dispersion statistics.

It therefore seems that (7.6), as well as being an important theoretical concept, is a feasible constraint to implement in practice and gives a model that is more acceptable than either (4.3) or (4.2*a*).

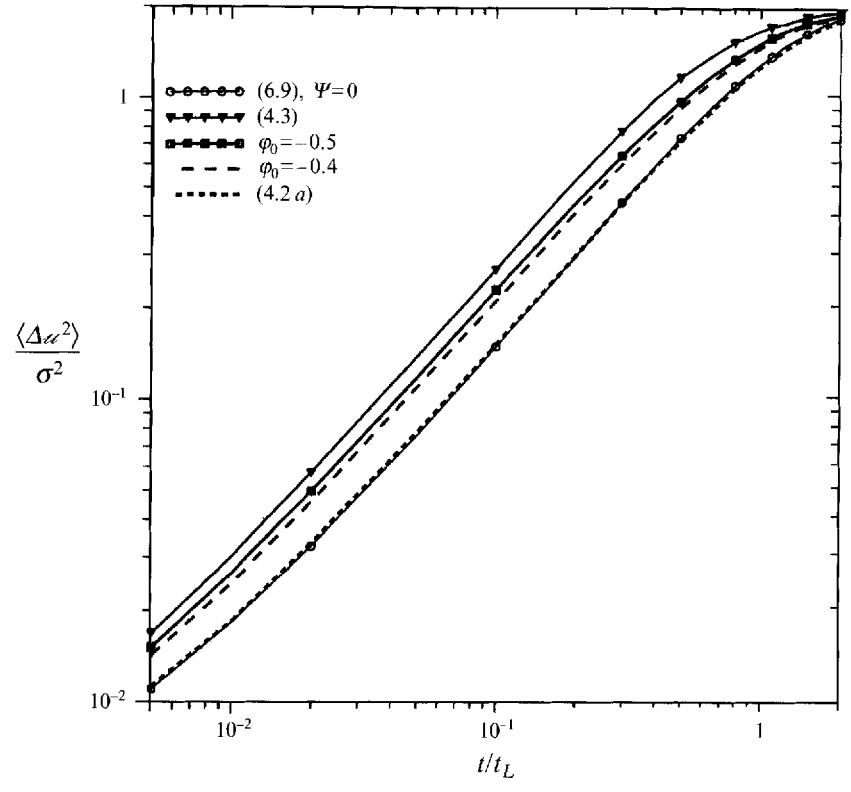

FIGURE 7. Relative velocity fluctuations, $\langle \Delta u^2 \rangle = \langle (u_i^{(1)} - u_i^{(2)})^2 \rangle$ where $u^{(1)} = u^{(1)} - u_0^{(1)}$ etc., is shown for the simulations described in figure 6. The inertial range is emphasized.

Relative-dispersion statistics from the new solutions are shown in figures 7 and 8 in which the inertial range is emphasized. The effect of varying φ_0 is evident by the quasi-parallel shift of the curves in these figures, which implies the functional dependence $\mathscr{C} = \mathscr{C}(\varphi_0)$ and $\tilde{\mathscr{C}} = \tilde{\mathscr{C}}(\varphi_0)$ of the inertial-range constants for velocity and displacement structure functions respectively. Now we note that the new solution with $\varphi_0 = -0.4$, which according to figure 6 is very nearly the 'best' solution based on the constancy of velocity variance, actually has inertial-range characteristics similar to those of model (4.3) ($\mathscr{C} \approx 7.5$ and $\tilde{\mathscr{C}} \approx 1.8$ when $C_0 = 4$). The results for the new model are $\mathscr{C} \approx 6$ and $\tilde{\mathscr{C}} \approx 1.3$, i.e. slightly smaller numerical values than for model (4.3) but not as small as Fung *et al.*s (1991) results (which also imply larger C_0). The differences for larger times (beyond the inertial range) are inconclusive because of the neglect of forcing terms. Thus in practical terms, model (4.3) and the more completely constrained model are not markedly different. The improved dynamics of the new model implies a reduction in the value of the constants from Thomson's model (4.3), in line with kinematic simulations, and perhaps a model which used more realistic (non-Gaussian) P_E and more complex (non-quadratic form) a would follow this trend.

8. The magnitude of C_0 and the diffusion limit

There is considerable debate about the value of the universal constant C_0, let alone the values of $\mathscr{C}$ and $\tilde{\mathscr{C}}$. The value $C_0 = 4$ falls in the middle of the experimental estimates by Hanna (1981), but recent analysis (Sawford 1991) indicates that a higher value may be appropriate. Here we explore the effects of varying C_0 on the relative-

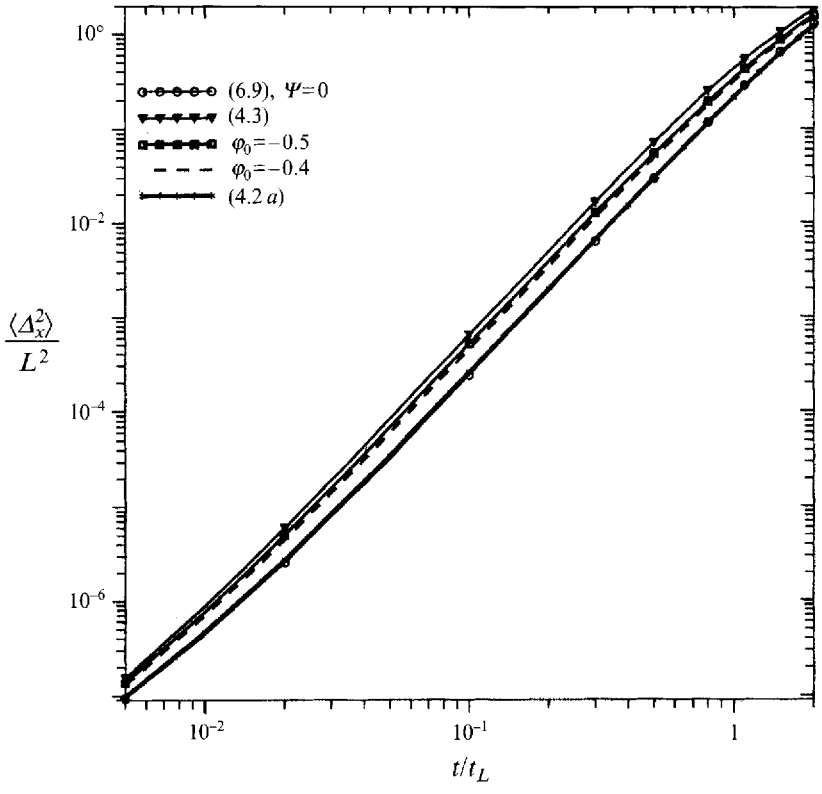

FIGURE 8. Mean-square relative displacement, $\langle \Delta_x^2 \rangle = \langle (x_i^{(1)} - x_i^{(2)})^2 \rangle$ where $x^{(1)} = \boldsymbol{x}^{(1)} - \boldsymbol{x}_0^{(1)} - \boldsymbol{u}_0^{(1)} t$ etc., is shown for the parameters of figure 6, emphasizing the inertial range.

dispersion inertial-range constant, $\tilde{\mathscr{C}}$ (see (4.5)), for some of the quadratic-form models derived in §§4 and 7. Figure 9 shows estimates of $\tilde{\mathscr{C}}$ obtained numerically from the slope of plots of $\langle \Delta^2 \rangle \approx \tilde{\mathscr{C}} t^3$ with log–log axes for several models. The diffusion limit, which corresponds to the limit $C_0 \to \infty$ and is derived in Appendix C, and the line $\tilde{\mathscr{C}} = 2C_0$ (for independent particles) are also shown. For large C_0 the numerical values for all models converge to the diffusion limit, confirming Thomson's (1987) theoretical analysis. However, the diffusion limit is a good approximation to relative dispersion (according to these quadratic models) only for $C_0 \geqslant 20$, which is much larger than the accepted range $C_0 \sim 2$–7. In general, the quadratic models that we have examined diverge in their relative-dispersion predictions for small C_0. However, over all the range of C_0 for which we have made calculations, Thomson's model (4.3) and the optimum two-to-one model (7.7) give similar relative-dispersion predictions in the inertial sub-range for moderate to large C_0, but differ for smaller (probably less relevant) C_0.

For $C_0 \leqslant 2$ relative dispersion according to both these models proceeds at a rate greater than that of independent particles (i.e. $\tilde{\mathscr{C}} > 2C_0$). This violates the inequality constraint derived by Borgas & Sawford (1991) who showed that the effect of correlation between the accelerations of the pair of particles is to reduce the relative dispersion.

9. Conclusions

In this paper we have shown explicitly that there is non-trivial non-uniqueness in two-particle Markovian stochastic models of dispersion, i.e. different models can be

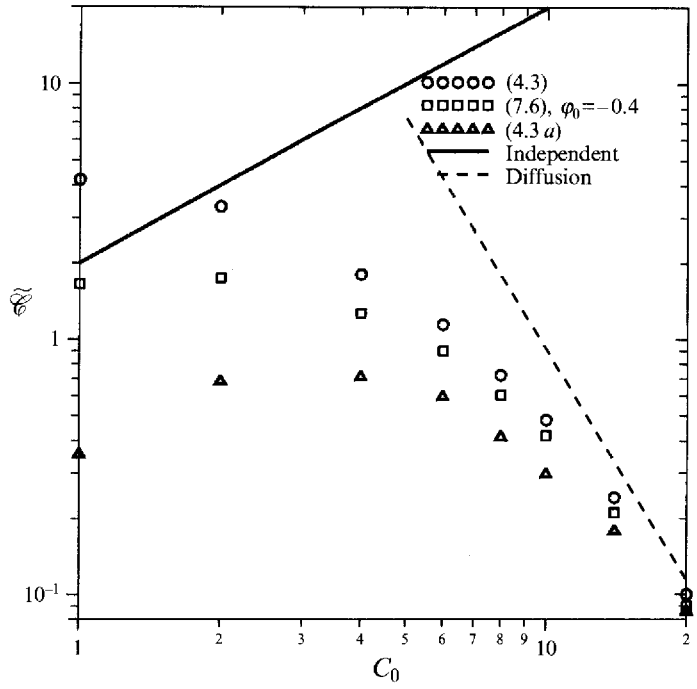

FIGURE 9. Inertial sub-range dispersion constant $\tilde{\mathscr{C}}$ as a function of C_0 for the reference model (4.2a). Thomson's model (4.3) and the new constrained model (7.6) with $\varphi_0 = -0.4$. The diffusion limit (Appendix C) and the independent-particle case are shown as solid lines.

obtained for the same Eulerian velocity statistics and such models have different Lagrangian statistics. In particular, the inertial-range dispersion and velocity statistics depend on the model chosen. We have also shown that the one-particle velocity variance, which is a known Lagrangian statistic, is not adequately modelled by every model.

A hierarchy of constraints has been developed so that two-particle statistics generated by the constrained stochastic models reduce to one-particle statistics when averaged over either one of the particles. The hierarchy is ordered by a power series in a time increment, δt, from an initial instant where the particles have fixed separation Δ_0. These constraints have been considered to second order in δt, and can be related to conditionally averaged Navier–Stokes equations. This represents a novel incorporation of explicit mechanics for such stochastic models. Although the constraints that are derived here have simple physical interpretations, our procedure shows them to be related to two-to-one reduction in a rational way.

The Navier–Stokes equations together with the Gaussian Eulerian statistics lead naturally to models which are quadratic in the particle velocities. A class of such quadratic-form models has been derived for isotropic homogeneous turbulence for Gaussian two-point (Eulerian) velocity statistics. When constrained only by symmetry and Thomson's (1987) well-mixed criterion, this class reduces to five unknown functions of Δ. Thomson's (1990) model is included in this class of models and calculations show that it generates the one-particle velocity variance to within the statistical sampling uncertainty despite having conditional accelerations inconsistent with the Navier–Stokes equations.

The quadratic-form models, when additionally forced to satisfy the second-order two-to-one reduction constraints, reduce to a family dependent on only one unknown

function of Δ. Numerical simulations for times larger than which a truncated δt-series expansion is accurate, have shown that not all of these constrained models generate satisfactory one-particle statistics. However, it is possible to choose the unknown function to optimize the model's prediction of the one-particle velocity variance. The one-particle performance of this optimal model is comparable with Thomson's (1990) two-particle model. The two-particle results from either Thomson's model or the newly constructed quadratic model are similar, although the conditional accelerations for each model clearly differ. This suggests that the dispersion described by these two models is fairly robust.

We have chosen the final unknown function in the class of quadratic models empirically (so that one-particle velocity variance and one-particle dispersion are approximately correct). In principle, further two-to-one reduction constraints (say at third order in δt) would fix this arbitrariness. However, except in Novikov's (1963) case of dual particle independence and for trivial two-point acceleration correlations, it is clear that there is insufficient freedom within quadratic solutions to impose all the constraints at higher order in δt.

A fully general analysis involves relaxation of both the assumption of Gaussian Eulerian statistics and the choice of quadratic drift terms. This seems formidable and perhaps unwarranted, especially because the newer more highly constrained model differs moderately from that of Thomson (1990) in quantities of practical interest.

We examined the variation of inertial sub-range constant for relative dispersion, $\tilde{\mathscr{C}}$, with the velocity-structure constant, C_0, through numerical simulations for three models: the new model, Thomson's model and the reference model. The new constrained model and Thomson's model are similar across the range of C_0 values considered (1–20) and differ most significantly for small C_0. However, there is a considerable difference between those two models and the reference model (by a factor of two or three in the physically meaningful range of C_0). Although the reference model can be discounted because it violates the second-order constraints and also because its one-particle statistics are relatively poor, the large differences suggest that the small-scale structure (the inertial range in particular) is sensitive to model details. Furthermore, the effect of uncertainty in C_0 is probably at least as significant as the differences between models.

Appendix A

In this Appendix we consider the Eulerian conditional average, $\tilde{\mathscr{P}} = \langle p^{(2)} \,|\, \boldsymbol{u}^{(1)} \rangle$, which is the mean pressure at $\boldsymbol{x}+\boldsymbol{\Delta}$ given that the velocity at $\boldsymbol{x}$ is $\boldsymbol{u}^{(1)}$. $\tilde{\mathscr{P}}$ is a function of both $\boldsymbol{u}^{(1)}$ and $\boldsymbol{\Delta}$ and can be determined from a Poisson equation (see Batchelor 1953):

$$\nabla^2 \tilde{\mathscr{P}} = -\rho \left\langle \frac{\partial u_i^{(2)}}{\partial \Delta_j} \frac{\partial u_j^{(2)}}{\partial \Delta_i} \,|\, \boldsymbol{u}^{(1)} \right\rangle = -\rho \frac{\partial^2}{\partial \Delta_i \partial \Delta_j} \langle u_i^{(2)} u_j^{(2)} \,|\, \boldsymbol{u}^{(1)} \rangle, \tag{A 1}$$

where $\nabla^2 \tilde{\mathscr{P}} = \partial^2 \tilde{\mathscr{P}} / \partial \Delta_i \partial \Delta_i$. Equation (A 1) can be solved for $\tilde{\mathscr{P}}$ provided that the tensor

$$\langle u_i^{(2)} u_j^{(2)} \,|\, \boldsymbol{u}^{(1)} \rangle$$

can be specified. This tensor can be determined directly from $P_E(\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}; \boldsymbol{\Delta})$, and in particular for a Gaussian form of this probability distribution we have

$$\langle u_i^{(2)} u_j^{(2)} \,|\, \boldsymbol{u}^{(1)} \rangle = \rho_{il} \rho_{jm} u_l^{(1)} u_m^{(1)} - \sigma^2 (\delta_{ij} - \rho_{ik} \rho_{kj}),$$

which is the form we shall use. The solution of (A 1) implies that

$$\tilde{\mathscr{P}}(\boldsymbol{u}^{(1)}, \boldsymbol{\Delta}) = \mathscr{P}_{ij}(\boldsymbol{\Delta}) u_i^{(1)} u_j^{(1)} + \sigma^2 \hat{\mathscr{P}}(\boldsymbol{\Delta}),$$

where $\mathscr{P}_{ij}(\Delta) = (\not{p}_1(\Delta) - \not{p}_2(\Delta))\frac{\Delta_i \Delta_j}{\Delta^2} + \not{p}_2(\Delta)\,\delta_{ij}$ and $\hat{\mathscr{P}}(\Delta) = \not{p}_3(\Delta)$.

The Poisson equation then reduces to a set of ordinary differential equations for the functions $\not{p}_1$, $\not{p}_2$ and $\not{p}_3$. The solutions so obtained are unique under the specification of the boundary condition at infinity, which is that the conditional pressure expectation vanishes for infinitely large separations.

The solution for the above conditions has

$$\not{p}_1 = -\tfrac{1}{6}\rho \int_\Delta^\infty \xi f'^2 \,\mathrm{d}\xi + \tfrac{2}{5}\rho\Delta^2 \int_\Delta^\infty \xi^{-1} f'^2 \,\mathrm{d}\xi + \tfrac{7}{30}\rho\Delta^{-3} \int_0^\Delta \xi^4 f'^2 \,\mathrm{d}\xi, \tag{A 2}$$

$$\not{p}_2 = -\tfrac{1}{6}\rho \int_\Delta^\infty \xi f'^2 \,\mathrm{d}\xi - \tfrac{1}{5}\rho\Delta_2 \int_\Delta^\infty \xi^{-1} f'^2 \,\mathrm{d}\xi - \tfrac{7}{60}\rho\Delta^{-3} \int_0^\Delta \xi^4 f'^2 \,\mathrm{d}\xi \tag{A 3}$$

and

$$\not{p}_3 = \tfrac{1}{2}\rho \int_\Delta^\infty \xi f'^2 \,\mathrm{d}\xi. \tag{A 4}$$

For the calculation of particle two's mean conditional acceleration we require the pressure gradient. Furthermore, we ignore the contribution from the viscous stress provided that $\Delta \gg (\nu^3/\epsilon)^{1/4}$ (the Kolmogorov microscale) and we ignore any contribution from external forcing, i.e. we assume that the forcing at particle two's position is independent of $\boldsymbol{u}^{(1)}$. In terms of our solutions the pressure gradient is expressed as

$$\left\langle \frac{\partial p^{(2)}}{\partial \Delta_i} \middle| \boldsymbol{u}^{(1)} \right\rangle = \Lambda_{ijk}\, u_j^{(1)}\, u_j^{(1)} + \sigma^2 \not{p}_3' \frac{\Delta_i}{\Delta}, \tag{A 5}$$

where

$$\Lambda_{ijk} = (\not{p}_1' - \not{p}_2' - \frac{2}{\Delta}(\not{p}_1 - \not{p}_2))\frac{\Delta_i \Delta_j \Delta_k}{\Delta^3} + \not{p}_2' \frac{\Lambda_i}{\Delta}\delta_{jk} + \frac{1}{\Delta}(\not{p}_1 - \not{p}_2)\left(\frac{\Delta_j}{\Delta}\delta_{ik} + \frac{\Delta_k}{\Delta}\delta_{ij}\right).$$

Because of the derivatives and combination of $\not{p}_1 - \not{p}_2$ in Λ_{ijk} the numerical calculation of (A 5) only requires the computation of

$$\int_\Delta^\infty \xi^{-1} f'^2 \,\mathrm{d}\xi \quad \text{and} \quad \int_0^\Delta \xi^4 f'^2 \,\mathrm{d}\xi,$$

which we denote by $\mathscr{I}_1^*(\Delta)$ and $\mathscr{I}_2^*(\Delta)$ respectively. This, in general, is a numerical task and requires the parameterization of the structure function $f(\Delta)$. We shall use Durbin's (1980) form (see also (3.3)),

$$f(\Delta) = 1 - \left(\frac{\Delta^2}{\Delta^2 + \mathscr{L}^2}\right)^{1/3}$$

where $\mathscr{L}$ is a lengthscale. In this case we have

$$f'(\Delta) = -\tfrac{2}{3}\mathscr{L}^2 \Delta^{-1/3}(\Delta^2 + \mathscr{L}^2)^{-4/3}$$

and so the integrals may be expressed as

$$\mathscr{I}_1^*(\Delta) = \tfrac{4}{9}\mathscr{L}^{-2} \int_{\Delta/\mathscr{L}}^\infty \xi^{-5/3}(1+\xi^2)^{-8/3} \,\mathrm{d}\xi$$

and

$$\mathscr{I}_2^*(\Delta) = \tfrac{4}{9}\mathscr{L}^3 \int_0^{\Delta/\mathscr{L}} \xi^{10/3}(1+\xi^2)^{-8/3} \,\mathrm{d}\xi.$$

The integrals in a non-dimensional form,

$$\mathscr{I}_1^*(\Delta) = \tfrac{4}{9}\mathscr{L}^{-2}\mathscr{I}_1\left(\frac{\Delta}{\mathscr{L}}\right) \quad \text{and} \quad \mathscr{I}_2^*(\Delta) = \tfrac{4}{9}\mathscr{L}^{3}\mathscr{I}_2\left(\frac{\Delta}{\mathscr{L}}\right),$$

can be tabulated, using a combination of both Simpson's rule and asymptotic expansions, to more than sufficient accuracy. Thus it is feasible to implement the conditional acceleration constraints in the stochastic model.

Appendix B

The expansion for particle one's velocity variance is

$$\langle u_{i'} u_{j'}\rangle = \sigma^2\delta_{i'j'} + \left(\frac{\partial}{\partial \Delta_{k''}}\langle u_{j'} u_{k''} a_{i'}\rangle + \frac{\partial}{\partial \Delta_{k''}}\langle u_{i'} u_{k''} a_{j'}\rangle\right)_{\Delta=\Delta_0}(t-t_0)^2 + \ldots, \tag{B 1}$$

where we are following the earlier conventions and have used (6.8). Using the representation of §5, i.e. assuming Gaussian P_E, we find that

$$\langle u_{j'} u_{k''} a_{i'}\rangle = \rho_{i'l''}\frac{\partial \rho_{j'k''}}{\partial \Delta_{l''}} + 2(\tilde{\gamma}^{\{1\}}_{i'j'l'}\rho_{l'k''} + \tilde{\gamma}^{\{3\}}_{i'l'm'}\rho_{l'j'}\rho_{m'k''} - \tilde{\gamma}^{\{4\}}_{i'j'k''} - \tilde{\gamma}^{\{2\}}_{i'l'k''}\rho_{l'j'}) \tag{B 2}$$

and a similar expression for the other part of (B 1). Finally, using the constraints (5.8) and (7.3), gives that, for the coefficient of $(t-t_0)^2$ in (C 1) to vanish, we must have

$$\begin{aligned}
(1-f^2)(1+f)\,\tilde{A}^{\{2\}} &= ff', \\
(1-g^2)(1+f)(\tilde{B}^{\{1\}} + \tilde{B}^{\{2\}}) &= -\tfrac{1}{4}gf', \\
-\tfrac{1}{2}(1-f)(\tilde{B}^{\{1\}} - \tilde{B}^{\{2\}}) + (1-fg^2)\,\tilde{C}^{\{2\}} - (f-g^2)\,\tilde{D}^{\{3\}} &= \tfrac{1}{2}fg' - \tfrac{1}{4}gf'.
\end{aligned} \tag{B 3}$$

Now (B 3) is actually exactly the consequence of (6.9), with zero $\boldsymbol{\Psi}$-vector. However, because of the gradient operator in (B 1), a non-trivial divergence-free $\boldsymbol{\Psi}$-vector will not alter the coefficient, so that (6.12) (or more properly (7.5)) will also imply that the one-particle velocity variance is constant at least to second order in $t-t_0$.

Appendix C

Thomson (1987) has shown that in the limit $t_L = 2\sigma^2/C_0\epsilon \to 0$, (2.1) has a diffusion limit in which particle position is Markovian. For a given flow (in which σ^2 and ϵ are specified) it is appropriate to identify this diffusion limit with the limit $C_0 \to \infty$. Thus we anticipate that this limit will determine the behaviour of the stochastic models under consideration here as C_0 is allowed to increase.

All of the models we have considered have been of the form

$$\mathrm{d}u_i = -\frac{1}{2}\frac{C_0\epsilon}{\sigma^2}\lambda_{ij}u_j\,\mathrm{d}t + \phi_i\,\mathrm{d}t + (C_0\epsilon)^{1/2}\,\mathrm{d}W_i \quad (i = 1,2,\ldots,6), \tag{C 1}$$

where ϕ_i is a quadratic function of $\boldsymbol{u}$ (see §4). Because of this quadratic behaviour, the diffusion limit is independent of the details of ϕ_i, i.e. all of the models we consider have the same diffusion limit (Thomson 1987). For a Gaussian P_E, this limit is

$$\mathrm{d}x_i = \frac{\partial \kappa_{ij}}{\partial x_j}\,\mathrm{d}t + \left(\frac{4\sigma^4}{C_0\epsilon}\right)^{1/2}\lambda^{-1}_{ij}\,\mathrm{d}W_j \quad (i = 1,2,\ldots,6), \tag{C 2}$$

where the diffusivity tensor is given by

$$\kappa_{ij} = \frac{2\sigma^4}{C_0 \epsilon} \lambda_{ik}^{-1} \lambda_{kj}^{-1} \quad (i,j = 1,2,\ldots,6) \tag{C 3}$$

and we recall that $\sigma^2 \lambda_{ij}^{-1}$ is the six-dimensional Eulerian velocity correlation tensor (see (3.6)) and is a function of Δ. Using the block structure (3.1) to write $\boldsymbol{\lambda}^{-1}$ in terms of one-point and two-point correlation tensors, and introducing the isotropic forms (3.1 *a–c*) and we can derive from (C 3) a diffusion equation for the particle separation, $\boldsymbol{\Delta} = \boldsymbol{x}^{(2)} - \boldsymbol{x}^{(1)}$,

$$\mathrm{d}\Delta_i = -\frac{4\sigma^4}{C_0 \epsilon} f'^2(\Delta)\, \Delta_i \, \mathrm{d}t + 2\left(\frac{4\sigma^4}{C_0 \epsilon}\right)^{1/2} \left((f-g) \frac{\Delta_i \Delta_j}{\Delta^2} + (g-1)\, \delta_{ij} \right) \mathrm{d}W_j \quad (i = 1,2,3). \tag{C 5}$$

The Fokker–Planck equation corresponding to (C 4) can be written

$$\frac{\partial P(\Delta, t; \Delta_0, 0)}{\partial t} = \frac{1}{\Delta^2} \frac{4\sigma^4}{C_0 \epsilon} \frac{\partial}{\partial \Delta} \left(\Delta^2 (1-f)^2 \frac{\partial P(\Delta, t; \Delta_0, 0)}{\partial \Delta} \right) \tag{C 5}$$

and for inertial-range separations, (3.3) can be used to write (C 5) as

$$\frac{\partial P(\Delta, t; \Delta_0, 0)}{\partial t} = \frac{C^2 \epsilon^{1/3}}{C_0} \left(\tfrac{10}{3} \Delta^{1/3} \frac{\partial P(\Delta, t; \Delta_0, 0)}{\partial \Delta} + \Delta^{4/3} \frac{\partial^2 P(\Delta, t; \Delta_0, 0)}{\partial \Delta^2} \right). \tag{C 6}$$

For travel times within the inertial sub-range, i.e. for $t_0 \ll t \ll t_L$, the initial dependence on Δ_0 disappears and P is a function only of the scalar separation. A similarity solution of (B 6) can then be found:

$$P(\xi) = \frac{2187}{560} \pi^{-1/2} \exp\left(-\tfrac{9}{4}\xi^{2/3}\right);$$

then with $\xi = \Delta (4C^2 \epsilon^{1/3} t / C)^{-3/2}$ we have the time-dependent solution for Δ-statistics.

Hence, in the diffusion limit, the mean-square separation in the inertial sub-range is

$$\langle \Delta^2 \rangle = \tfrac{1144}{81} \frac{C^6}{C_0^3} \epsilon t^3 \quad (t_0 \ll t \ll t_L).$$

Thus in this limit the inertial sub-range constant $\tilde{\mathscr{C}}$ from (4.5) is given by

$$\tilde{\mathscr{C}} = \frac{1144}{81} \frac{C^6}{C_0^3}.$$

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